61. Show that the TSP is a special case of the QAP.

62. Formulate the question whether two given undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|V_1| = |V_2|$ are isomorphic, as a QAP.

63. Consider the following graph packing problem

    Given: Undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|V_1| = |V_2| = n$.
    Task: Find a packing of G_2 into G_1 if one exists.

    A packing of G_2 into G_1 is a permutation π of $\{1, \ldots, n\}$ such that

    $$(i, j) \in E_1 \Rightarrow (\pi(i), \pi(j)) \notin E_2$$

    Formulate the graph packing problem as QAP.

64. Consider the following situation: n persons are invited to a wedding dinner. They have to be seated around a round table. Each person can easily communicate with its left and its right neighbour and with a higher effort it is also possible to talk to their neighbours to the left and right, respectively. The hosts know (or guess) a coefficient a_{ik} which measures the pleasure that person i gets from communicating with person k. They want to place the guests such that the total pleasure is maximized.

    Model this as a QAP. In which way does the shape of the table influence the model? What would change for a square or long rectangular table?

65. Prove that the Lawler linearization introduced in the lecture is equivalent to the QAP.

66. Prove that the Kaufman Broeckx linearization introduced in the lecture is equivalent to the QAP.

67. Consider a QAP in Koopmans Beckman form. Provide a formula for the mean of the objective function value (over all possible permutations). Do the same for the Lawler QAP.

68. Consider the following instance of the QAP in Koopmans Beckmann form (with $C = 0$):

    $$A = \begin{pmatrix} 2 & 3 & 4 & 0 \\ 3 & 0 & 5 & 7 \\ 8 & 2 & 1 & 7 \\ 0 & 0 & 4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 8 & 3 & 1 \\ 0 & 4 & 2 & 7 \\ 0 & 0 & 3 & 7 \\ 2 & 1 & 0 & 7 \end{pmatrix}$$

    Compute the Gilmore-Lawler bound (GLB). Consider both the variant with diagonal entries as well as the variant which shifts the pure quadratic terms into the linear term before the bound is calculated.

69. Compute the Gilmore-Lawler bound for QAP instance from Problem 68 under the additional constraint that $\varphi(1) = 2$ and $\varphi(2) = 3$ holds (i.e., the elements 1 and 2 are already assigned).

70. Devise a branching strategy for the QAP where in each iteration one additional component of the assignment gets fixed.

71. Combine the ideas from the previous exercises and the lecture to obtain a branch & bound approach for the QAP and apply it to the instance from Problem 68.

72. The quadratic bottleneck assignment problem (BQAP) results from the QAP if the sum terms in the objective function are replaced by maxima. The BQAP in Koopmans Beckmann looks as follows

$$\min_{\varphi \in \mathcal{S}_n} \max \left(\max_{i,k=1,\ldots,n} a_{ik} b_{\varphi(i)\varphi(k)}, \max_{i=1,\ldots,n} c_{i\varphi(i)} \right)$$

Describe how the idea of the Gilmore-Lawler bounds can be carried over to the bottleneck case.

73. Apply your approach from Problem 72 to the BQAP instance with $C = 0$ and

$$A = \begin{pmatrix} 7 & 2 & 1 & 7 \\ 0 & 0 & 4 & 2 \\ 3 & 0 & 5 & 6 \\ 2 & 3 & 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 7 & 3 & 1 \\ 2 & 1 & 0 & 6 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 3 & 7 \end{pmatrix}$$

74. Develop an efficient algorithm for the special case of the quadratic bottleneck assignment problem in Koopmans Beckmann form which results when the matrices A and B are symmetric, nonnegative $n \times n$ product matrices.

75. Develop an efficient algorithm for the special case of the quadratic bottleneck assignment problem in Koopmans Beckmann form which results when the matrix A is an $n \times n$ circulant with nonnegative entries and B is an $n \times n$ matrix with $b_{ij} = \max\{u_i, v_j\}$ where $u, v \in \mathrm{I\!R}^n$.

76. We are given an undirected graph $G = (V, E)$, edge weights c_e for the edges $e \in E$ and a number $k \in \mathrm{I\!N}$. Show that the problem of determining a matching with minimum weight among all matchings with exactly k edges can be reduced to the problem of finding a minimum weight perfect matching.

77. Let an instance (G, c) of the minimum weight perfect matching problem be given. Show that the dual solution (y, Y) obtained by setting $y_v := \min\{\, c_e/2 : e \in \delta(v) \,\}$ for all $v \in V$ and $Y_D = 0$ for all odd cuts D is feasible for the dual problem and fulfills complementary slackness together with the empty matching $M = \emptyset$. (This solution can be used as starting solution.)

78. Use the blossom algorithm to determine a minimum cost perfect matching and an optimal solution of the associated dual problem for the matching instance obtained from the complete graph on 6 vertices with the following edge weights: $c_{12} = 30$, $c_{13} = 38$, $c_{14} = 36$, $c_{15} = 2$, $c_{16} = 58$, $c_{23} = 4$, $c_{24} = 28$, $c_{25} = 28$, $c_{26} = 80$, $c_{34} = 68$, $c_{35} = 40$, $c_{36} = 14$, $c_{45} = 18$, $c_{46} = 70$, $c_{56} = 36$.

79. Use the blossom algorithm to determine a minimum cost perfect matching and an optimal solution of the associated dual problem for the matching instance depicted in Figure 1.

80. Given an undirected graph $G = (V, E)$ and edge weights $c_e \in \mathrm{I\!R}$ for $e \in E$, the *bottleneck perfect matching problem* asks for a perfect matching M which minimizes the bottleneck weight $b(M) = \max_{e \in E} c_e$. Derive an efficient algorithm for this problem and analyse its running time.

81. In the *minimum weight edge cover problem* we are given an undirected graph $G = (V, E)$ and edge weights $c_e \in \mathrm{I\!R}$ for $e \in E$. The task is to find an edge cover F with minimum cost $c(F) = \sum_{e \in F} c_e$. ($F \subseteq E$ is an edge cover if every vertex in V is incident with at least one edge from F).

    Derive an efficient algorithm to solve this problem.

82. Provide a (counter)example that demonstrates that the dual of the LP formulation of the minimum weight perfect matching problem need not have an integral optimal solution even when all edge weights are integral.

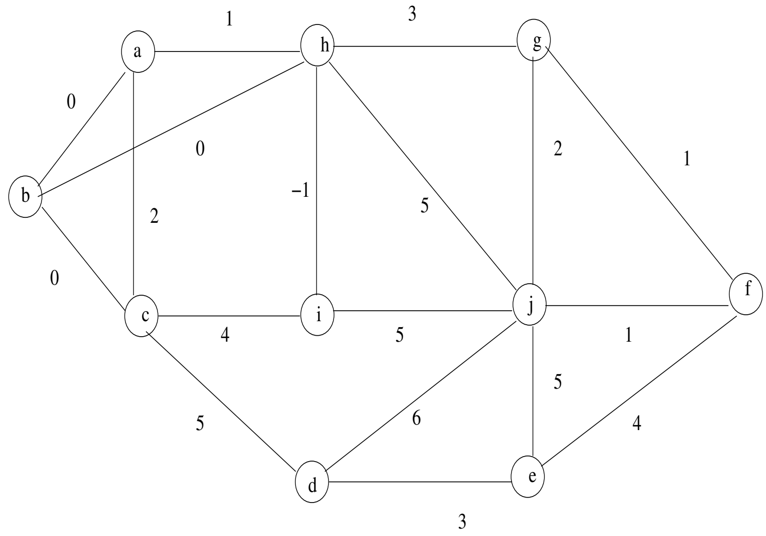

Figure 1

83. Prove the part of Theorem 6.1 which has not been shown in the lecture, i.e. prove that G contains a perfect matching if and only if the linear program provided for the minimum weight perfect matching problem has a feasible solution. (Hint: Use LP duality theory and the blossom algorithm.)

84. Given an undirected graph $G = (V, E)$ with nonnegative edge weights c_e for $e \in E$ and two vertices $s, t \in V$, the task is to find a shortest s-t path with an even respectively an odd number of edges.

    Provide an efficient algorithm for these two variants of the undirected shortest path problem. (Hint: Take two copies of G, link the copies of the same vertex with 0-weight edges, delete s and t and solve an appropriate matching problem in the new graph.)

85. Provide an LP formulation for the maximum weight matching problem and prove its correctness.

86. Solve the instance of the Chinese postman problem depicted in Figure 1 algorithmically.

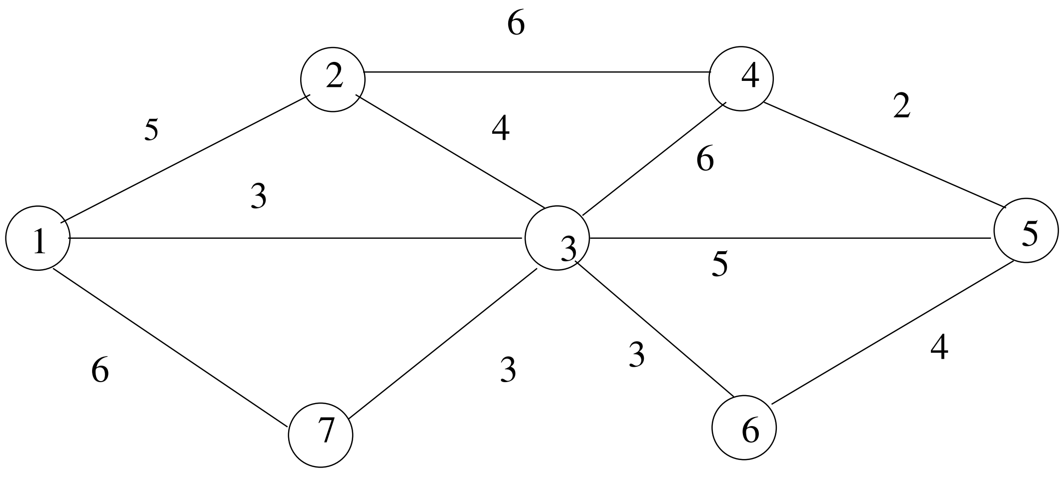

Figure 1

87. Solve the instance of the Chinese postman problem from Figure 2 algorithmically. Which edges are visited more than once in the optimal solution?

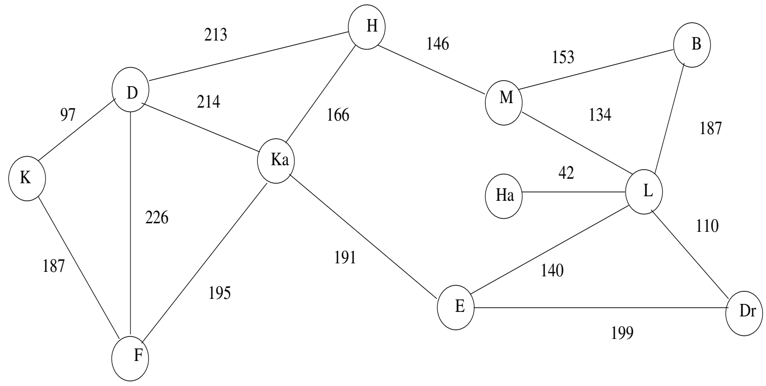

Figure 2

88. In the lecture we derived an algorithm for the Chinese postman problem in undirected graphs which is based on matching methods. Let us consider the version of the problem in directed graphs.

 (a) Does there always exist a postman tour in the directed case? If so, why? If not, under which additional conditions?

 (b) Develop an algorithm for finding a minimum cost (directed) Postman tour in directed graphs. (Hint: Flow problem)

 (c) Apply your algorithm to the example from Figure 3.

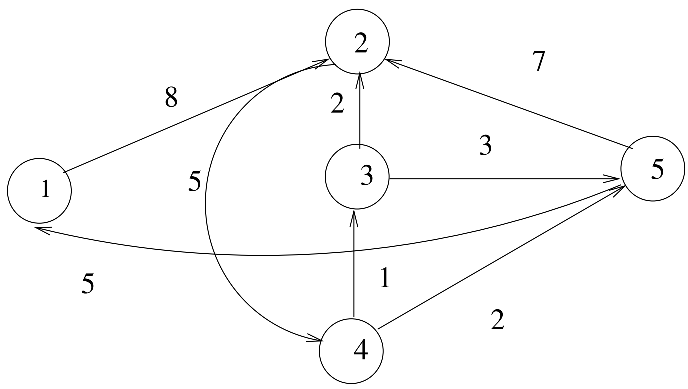

Figure 3