115. Decide whether the matrix A given below is totally unimodular.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

116. Prove the following statements

(a) If the matrix A is totally unimodular, then also the matrices A^t, $-A$, $(A|A)$, $(A|I)$ und $(A|-A)$ are totally unimodular (I denotes the identity matrix of the appropriate dimension).

(b) Provide a counterexample that demonstrates that the total unimodularity of the matrices A and B is not sufficient to guarantee that the composed matrix $(A|B)$ is totally unimodular.

117. Consider the following 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Is the matrix A totally unimodular?

(b) Prove that for all integer vectors $b \in \mathbb{R}^3$ for which $P(b) = \{x \in \mathbb{R}^3 : \mathrm{Ax} = \mathrm{b}\}$ is not empty all extreme points of $P(b)$ are integral.

(c) What is your explanation with respect to the results from (a) and (b) in light of Theorem 9.1 from the lecture? Is there something wrong?

118. Consider the directed graph $G = (V, A)$ displayed in Figure 21.

(a) Let T_1 be the directed tree that contains the arcs e_1, e_2 and e_3 of G. Obtain the network matrix D_1 which corresponds to G and T_1.

(b) Let T_2 be the directed tree that contains the arcs e_1, e_3 and e_5 of G. Obtain the network matrix D_2 which corresponds to G and T_2.

(c) Which (algebraic) connection exists between D_1 and D_2?

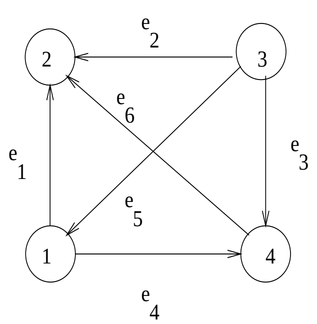

Figure 21

119. Let $G = (V, E)$ be an undirected graph. The vertex-edge incidence matrix associated with G is a 0-1 matrix that has a row for every vertex and a column for every edge. The column corresponding to the edge $\{i, j\}$ has a one in the row for vertex i and a one in the row for vertex j and zeros in all other rows.

(a) Show that the vertex-edge incidence matrix is not necessarily totally unimodular.

(b) Prove that the vertex-edge incidence matrix of bipartite graphs is totally unimodular.

120. An $m \times n$ matrix with 0-1 entries is called *interval matrix* if for each column the 1-entries form an interval, i.e. that $a_{ij} = a_{kj} = 1$ with $k > i + 1$ implies $a_{\ell j} = 1$ for all $\ell = i + 1, \dots, k - 1$.

(a) Prove that interval matrices are totally unimodular.

(b) Prove that interval matrices belong to the class of network matrices.

121. (For ambitious students) Prove the theorem of Ghoulia-Houri: Let A be an $m \times n$ matrix. The following two statements are equivalent to each other.

(a) A is totally unimodular.

(b) For *each* subset J of $\{1, \dots, n\}$ there exists a partition J_1, J_2 of J such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1$$

for all $i = 1, \dots, m$.

122. Prove that the theorem of Heller and Tomkins which has been proved in the lecture follows from the statement from Problem 121.

123. Consider the polytope $P := \{(x_1, x_2) \in \mathbb{R}^2 : \mathrm{x}_2 = \sqrt{2}\mathrm{x}_1, \mathrm{x}_1 \geq 0\}$. Prove that $P \neq P_I$.

124. Consider the two systems I: $x_1 + x_2 \leq 0, x_1 - x_2 \leq 0$ and II which results from adding $x_1 \leq 0$ to I. Prove that II is TDI and I is not TDI.

125. Consider the inequality system $-x_1 \leq 0$, $-x_2 \leq 0$, $x_1 + 2x_2 \leq 6$, $2x_1 + x_2 \leq 6$.

(a) Show that this system is not TDI.

(b) Obtain a TDI system which describes the same polyhedron by adding additional constraints to the system above.

126. Let a be an integral vector and β be a rational number. Prove that the inequality $a^t x \leq \beta$ is TDI if and only if the components of a are relatively prime.