101. Let $G = (V, E)$ be a given undirected graph and let M_1 and M_2 be two maximal matchings in G. Prove that $|M_1| \leq 2|M_2|$.

102. We are given an $n \times n$ matrix C with $0-1$ entries. Let $I = \{1, \ldots, n\}$ denote the set of rows and $J = \{1', \ldots, n'\}$ denote the set of columns of C. A *cover* of C is a set $K \subset I \cup J$ of rows and columns ($I = \emptyset$ or $J = \emptyset$ is allowed) such that whenever $c_{ij} = 1$ we have $i \in K$ or $j' \in K$, i.e., that the rows and columns in K cover all 1-entries of C.

    The task is to find a cover K with minimum cardinality.

    (a) Formulate this problem as an optimization problem in graphs dealt with in class.

    (b) Which other problem is closely related to the given problem? (Hint: First solve part (a).)

103. Consider the bipartite graph displayed in Figure 18. Determine a maximum matching by the following two approaches

    (a) transformation to a max flow problem

    (b) applying the matching algorithm (simpler for the bipartite case) presented in the lecture.

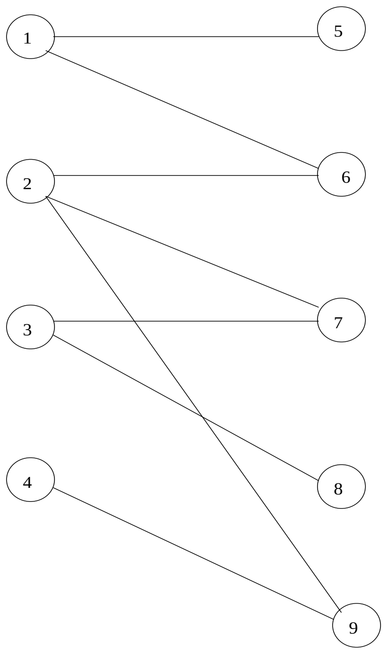

Figure 18

104. The permanent of an $q \times q$ matrix M is defined as

$$\text{per}(M) := \sum_{\pi \in S_q} \prod_{i=1}^{q} m_{i\pi(i)}$$

where S_q denotes the set of all permutations over $\{1, \ldots, q\}$. Let $G = (V, E)$ be a bipartite graph with bipartition $V_1 \cup V_2$ where $V_1 = \{i_1, \ldots, i_r\}$ and $V_2 = \{j_1, \ldots, j_r\}$.

Prove that the number of perfect matchings (=matchings with no free vertices) in G can be obtained by computing the permanent of the $r \times r$ 0-1 matrix M_G with $m_{st} = 1$ if $\{i_s, j_t\} \in E$ and $m_{st} = 0$ else.

105. In the *minimum edge cover problem* we are given an undirected graph $G = (V, E)$ with n vertices (all of degree ≥ 1). An edge cover is a subset $U \subseteq E$ of edges such that each vertex in V is incident with at least one edge from U. The task is to find an edge cover of minimum cardinality (=minimum edge cover).

    (a) Let U be a minimum edge cover. What can be said about the components of the subgraph of G which is induced by the edges in U? Which relation holds between the cardinality u of U and the number of components of the induced graph?

    (b) Explain how a matching M with $n - u$ edges can be derived from the edge cover U.

    (c) Explain how one can easily obtain an edge cover with at most $n - g$ edges starting from a matching in G with g edges.

    (d) Is there any relation between the smallest number u of edges in an edge cover and the largest number g of edges in a matching?

106. Prove the following statement: If the naive method for finding an augmenting path described in the lecture arrives at a parity conflict, the underlying graph is not bipartite. Is the converse of the statement true as well?

107. Let M_1 and M_2 be two arbitrary matchings in a bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$. Show that there exists a matching M which matches all vertices in V_1 matched by M_1 and all vertices in V_2 matched by M_2. (Hint: $M_1 \oplus M_2$ might help you.)

108. In the bin packing problem we are given n items. Item i has size $s_i \in (0, 1]$. The goal is to pack the items in the smallest number of containers of unit capacity.

    The bin packing problem is difficult in general. Show that the special case where all items have sizes $s_i > 1/3$ is much easier to handle.

109. Let $G = (V, E)$ be an undirected graph. An edge coloring in G is a coloring of the edges of E such that no two edges which share a common vertex have the same colour.

    (a) Prove that a k-coloring (edge coloring with k colours) in G defines a partitioning of the edge set E into k pairwise disjoint matchings.

    (b) A graph G is called k-regular if all vertices have degree k. Show that a k-regular bipartite graph always has a k-coloring.

    (c) Let $\delta(G)$ denote the maximum vertex degree in G. Prove that every bipartite graph has a $\delta(G)$-coloring.

110. A subset of vertices $V' \subseteq V$ is called *matchable* if there exists a matching M such that all vertices from V' are matched. Show that for each matchable set $V' \subseteq V$ there exists a maximum matching M^* which matches all vertives in V'.

111. Let $G = (V, E)$ be an undirected graph and M be a matching in G. Furthermore, let B be a blossom with respect to M. Let G^c denote the graph which results from G by contracting the blossom B.

    Professor May B. Right claims to have found a counterexample to the following claim: The contracted graph G^c contains an augmenting path P^c if and only of the original graph G contains an augmenting path P.

    The example of Prof. Right is shown in Figure 19 dargestellt. The left part shows the original graph G and the right part shows the graph G^c after the contraction. Matching arcs are shown in bold. The vertices $2, 3, 5$ define a blossom B in G. The graph G^c obtained from contracting B does not contain an augmenting path even though G contains an augmenting path.

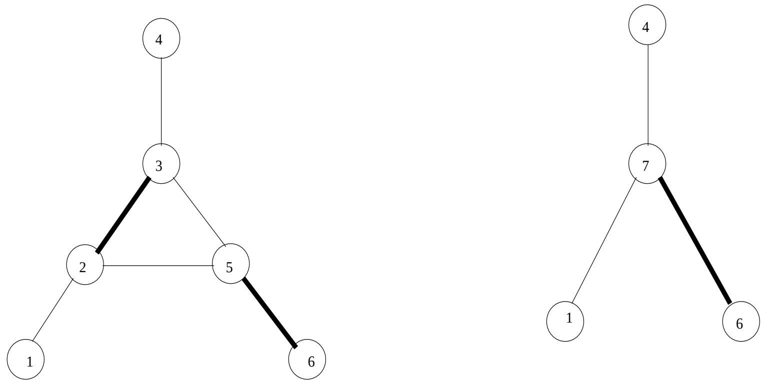

Figure 19

Has Prof. Right found a counterexample or has she made a mistake and if so which one?

112. Consider the graph given in Figure 20. Find a maximum matching starting from the initial matching which is shown in bold.

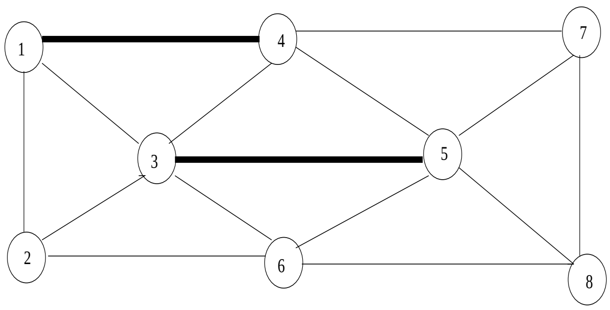

Figure 20

113. A ski instructor has n pair of skis with lengths $l_1 \leq l_2 \leq \ldots \leq l_n$ available to assign to n novice skiers. Suppose that the skiers are sorted with respect to their height, i.e., suppose $h_1 \leq h_2 \leq \ldots \leq h_n$. The length of the skis assigned to a skier should be proportional to the person's height. Let α be the constant of proportionality.

    The task of the instructor is to assign each skier i to a pair of skies $\pi(i)$ such that the total sum $\sum_{i=1}^{n} |l_i - \alpha h_{\pi(i)}|$ of differences between the optimal ski length and the assigned ski length is minimized.

    Formulate this problem as weighted matching problem in a bipartite graph and show that the matching which assigns skiier i the i-th pair of skis is an optimal solution.

114. (The stable matching problem) We are given a group which consists of n men and n women who want to marry. Each person ranks all group members of the other gender according to personal preferences. Let M be a perfect matching, i.e. n pairs of woman-man.

    A pair $(i, j) \in M$ consisting of a man i and a woman j is called instable if there exist a woman $j' \neq j$ and a man $i' \neq i$ such that $(i', j') \in M$ and man i prefers woman j' over his current partner j and woman j prefers man i' over her current partner i. A perfect matching M^* is called stable if it does not contain any instable pairs.

    (a) Does there always exist a stable matching?

(b) If there exists a stable matching, is it always unique?

(c) Derive an algorithm for obtaining a stable matching and prove its correctness.