85. Consider the directed graph $G = (V, A)$ shown in Figure 13 with source 1 and sink 5. The two arc weights attached to each arc (i, j) provide the capacity u_{ij} and the cost c_{ij} (in this order). Furthermore each vertex is associated a potential π_i, $i \in V$ (shown next to the vertices in the figure). Determine a flow of value 5 from s to t with minimum cost without using a min cost flow algorithm.

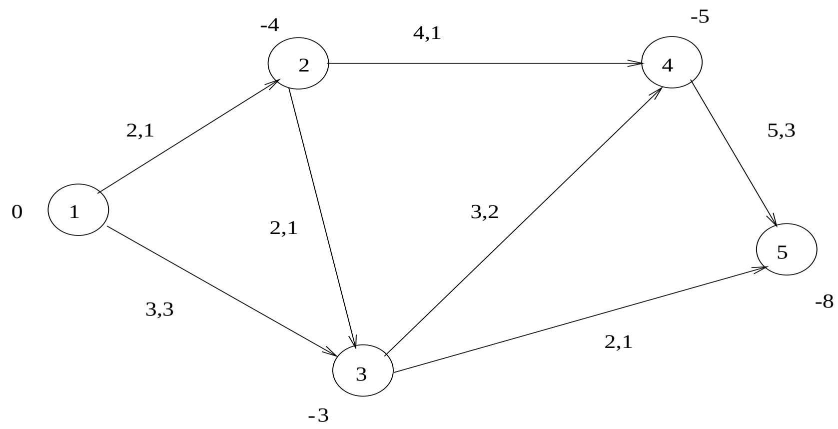

Figure 13

86. Consider the graph $G = (V, A)$ with source $s = 1$ and sink $t = 8$ shown in Figure 14. The two arc weights attached to each arc (i, j) provide the capacity u_{ij} and the cost c_{ij} (in this order).

   (a) Determine a flow of value 5 from s to t with minimum cost using an algorithm of your choice. Furthermore obtain an optimal solution of the dual problem.

   (b) What is the value of a maximum flow? Determine a minimum cut (= s-t cut).

   (c) Determine the cost curve $F(v)$ of a minimum cost flow of value v from s to t for all values v with $0 \leq v \leq 9$.

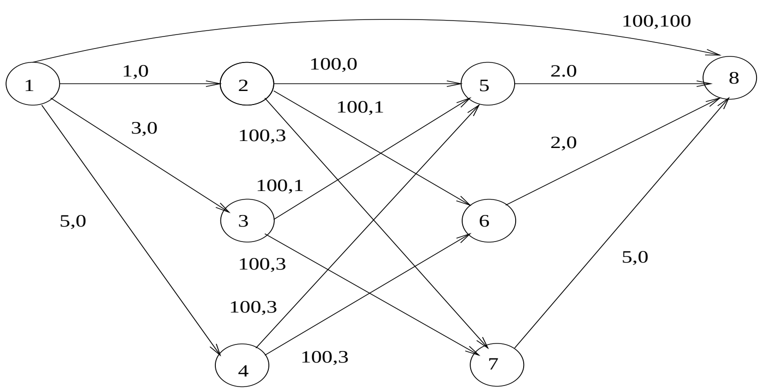

Figure 14

87. Determine a feasible flow for the instance minimum cost flow problem displayed in Figure 15 or argue that no such flow exists. The two values attached to each arc $(i, j) \in A$ respresent its lower capacity ℓ_{ij} and its upper capacity u_{ij}. The number attached to each vertex i represents its supply/demand value b_i.

88. Model the following problem as a network flow problem: We consider a time horizon of N days. Airline XYZ needs exactly r_j clean napkins on day j, $j = 1, \ldots, N$. There exist two alternatives to get clean napkins, either one can buy new napkins (instantly available for the day when they

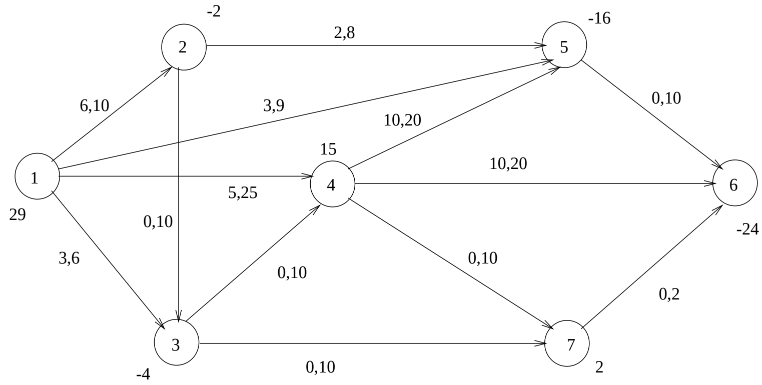

Figure 15

are bought) or have used ones cleaned by a laundry shop. There are two laundry modes: fast and normal. The fast mode takes $m > 0$ days, the normal mode n days where $n > m$. The price for a new napkin is $\bar{p}$ Cent, the normal laundry mode costs $\bar{s}$ Cent per napkin and the fast mode costs $\bar{q}$ Cent per napkin. How should the airline proceed so as to fulfill their demand for napkins on each of the N days such that the overall costs are minimized. Assume that the airline has no napkins in stock in the morning of day 1. Illustrate your ideas by a small example.

89. Consider the minimum cost flow problem with the following additional constraint: For each vertex $i \in V$ we are given an upper bound w_i for the amount of flow which enters vertex i. How can this version be transformed to the classical minimum cost flow problem?

90. Prove or disprove: If an instance of the minimum cost flow problem (MCF) on the graph $G = (V, A)$ has a feasible solution and if the cost function $\sum_{(i,j)\in A} c_{ij}x_{ij}$ is bounded from below over the set of feasible flows (e.g. always the case for bounded lower and upper capacities), then there exists a minimum cost flow x such that there does not exist a cycle Q in G for which $0 < x_{ij} < u_{ij}$ holds for all $(i, j) \in Q$. Which role do the assumptions play?

91. Suppose that we are given a rectangular $m \times n$ table with entries a_{ij} and we want to round each entry a_{ij} either to $\lfloor a_{ij} \rfloor$ or to $\lceil a_{ij} \rceil$. The goal is to perform the rounding such that the column and row sums of the rounded values change by less than one with respect to the original column and row sums.

    (a) Model the table rounding problem as finding a feasible flow in an appropriately defined graph with appropriately defined lower and upper capacities.

    (b) Demonstrate your approach for the following small example

3,7	6,8	7,3	17,8
9,6	2,4	0,7	12,7
3,6	1,2	6,5	11,3
16,9	10,4	14,5	

    (c) Now suppose that one does not want to find an arbitary feasible rounding, but one that minimizes the sum of squared errors (summed over all table entries) where the squared error obtained by rounding a_{ij} to $\bar{a}_{ij} \in \{\lfloor a_{ij} \rfloor, \lceil a_{ij} \rceil\}$ is given by $(\bar{a}_{ij} - a_{ij})^2$).
    How can one find a feasible rounding which minimizes this error (cost) term?

92. Consider the graph in Figure 16 with source 1 and sink 5. The three numbers attached to each arc (i, j) represent its cost c_{ij}, capacity u_{ij} and flow x_{ij} (in this order). Check whether the flow x is a maximum flow and and if so, if it is a maximum flow with minimum cost.

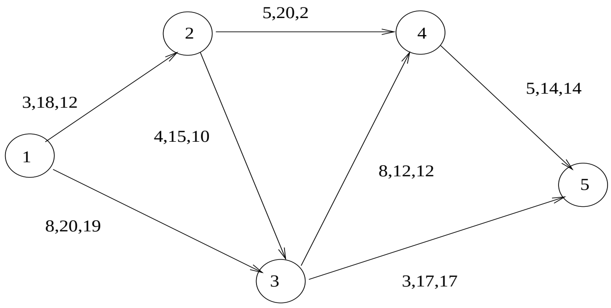

Figure 16

93. Consider the graph shown $G = (V, A)$ shown in Figure 17. The two numbers attached to each arc $(i, j) \in A$ represent its capacity u_{ij} and its cost c_{ij} (in this order).

   (a) Determine a maximum flow with minimum cost with the shortest augmenting path method and provide its cost.

   (b) Provide a flow with minimum cost among all flows from s to t with value 4.

   (c) Solve the task from (a) with the negative cycle method.

   (d) Solve the task from (a) with the cost scaling algorithm of Goldberg und Tarjan (tiresome manually).

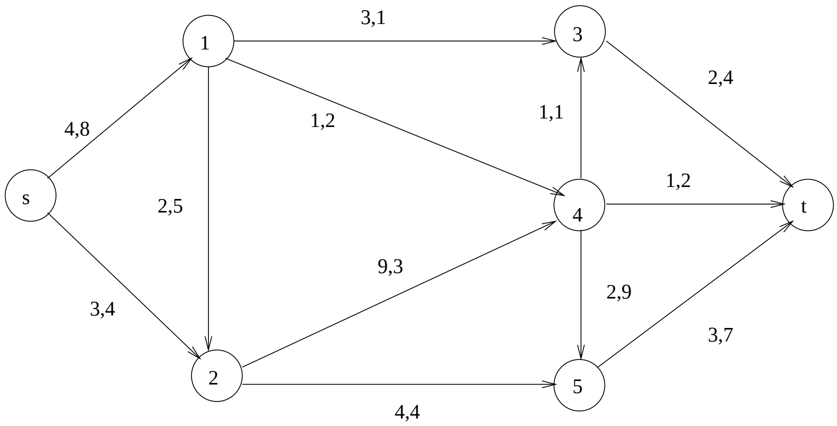

Figure 17

94. We are given a directed graph $G = (V, A)$. For each arc (i, j) we are given two numbers: its upper capacity u_{ij} and its cost c_{ij}. For each vertex i we are given its demand/supply b_i. Suppose that x is a minimum cost flow for the given instance.

   (a) Provide an efficient algorithm for determining a new minimum cost flow starting from x if the cost of a single arc (i, j) increases/decreases by 1.

   (b) Provide an efficient algorithm for determining a new minimum cost flow starting from x if b_k of a single vertex k increases by 1 and at the same time b_j of another vertex j descreases by 1.

   (c) Provide an efficient algorithm for determining a new minimum cost flow starting from x if the capacity of a single arc (i, j) increases by 1.

   (d) As in (c) for the case when an arc capacity $u_{ij} \geq 1$ is decreased by 1.

95. The airline "Happy Austria" offers 6 daily flights from Graz to Vienna. From 10 am to 8 pm every two hours a flight starts. The first three flights are carried out with an aircraft with a capacity of 100

passengers, the remaining flights can accomodate 150 passengers per flight. In case of overbooking, compensation payments are offered which amount to 50 Euro plus 10 Euro per hour of delay caused if a passenger is moved to a later flight. Passengers that do not get a seat in the 8pm flight can be handed over to another airline that offers a flight at 11pm with unlimited capacity (they use several planes if necessary). Suppose that the booking situation for the next day for the six available flights (in this order) is as follows: 110, 160, 103, 149, 175 and 140 bookings. For simplicity we assume that there is only one booking class.

Model this problem as a network flow problem.

96. A library which suffers from lack of primary storage space is considering to use secondary facilities, such as closed stacks or remote locations, to store parts of the books. There are q secondary facilities available and facility j has capacity b_j, $j = 1, \dots, q$. Let u_j be the access cost per book deposited in facility j. The books are grouped into p classes where class i consists of a_i books. Let u_i be the expected rate (number of books per unit time) that we will need to retrieve books from class i.

    The task is to develop a storage strategy which minimizes the (expected) retrieval cost.

    (a) Formulate this problem as minimum cost flow problem. Which specific structure does this problem have?

    (b) Show that the simple rule that repeatedly assigns items with the greatest retrieval rate to the storage facility with lowest access cost specifies an optimal solution.

97. We are given an undirected complete graph $G = (V, E)$ with arc costs c_{ij}, $(i, j) \in E$. Furthermore we are given a central vertex $s \in V$ and for each other vertex $i \in V \setminus \{s\}$ its demand r_i (a nonnegative integer). Further let R be a given positive integer.

    The task is to find a spanning tree T with minimum cost (=sum of the arc costs) such that the following condition is fulfilled: If we send flow from the central vertex s along the arcs in T to the other vertices so as to fulfill the demand r_i in every other vertex, then no arc receives a flow of more than R units of flow.

    Formulate this problem as a flow problem for the special case $r_i \in \{0, 1\}$ for all i and $R = 1$. Does this approach also work for the general case? If so, why? If not, why not?

98. We are given an instance of the minimum cost flow problem. An arc $(i, j) \in A$ is called *critical* if increasing resp. decreasing c_{ij} leads to an increase resp. descrease of the cost of a minimum cost flow.

    (a) Does there always exist a critical arc?

    (b) Develop an efficient algorithm for determining all critical arcs.

99. We are given a directed graph $G = (V, A)$ with arc capacities $u_{ij} \in \mathbb{R}_0^+$, $(i, j) \in A$, arc costs $c_{ij} \in \mathbb{R}_0^+$, $(i, j) \in A$ and a source $s \in V$ and a sink $t \in V$. We assume that $c(P) > 0$ (=sum of arc costs along the path) holds for all directed s-t paths P. Moreover a budget $B \in \mathbb{R}^+$ is given.

    The task in the budget constrained maximum flow problem (BCMFP) is to find a flow x from s to t in G which has the maximum value among all flows with total cost $\sum_{(i,j) \in E} x_{ij} c_{ij} \leq B$.

    Establish a connection between this problem and an appropriately chosen minimum cost flow problem and use this connection to determine a polynomial time algorithm for the BCMF for the case of integer capacities and costs. (Hint: Binary search.)

100. Consider a linear program of the form $\min c^t x$ s.t. $Ax \leq b, x \geq 0$ where A is a $p \times q$ 0-1 matrix satisfying the property that all of the 1's in each column appear consecutively (i.e., with no intervening zeros). Show that this special linear program can be transformed into a minimum cost flow problem.