64. Use the max flow min cut theorem by Ford and Fulkerson to prove the following theorem of Menger:

 The maximum number of arc-disjoint directed paths from a vertex s to a vertex t in a directed connected graph G is equal to the minimum number of arcs that need to be removed from G in order to destroy all directed paths from s to t.

65. Given a directed graph $G = (V, A)$ with a source s and a sink t, the task is to find a cut $(X, \overline{X})$ with $s \in X$, $t \in \overline{X}$ and minimal number of arcs $(=|\{(i,j) \in A : i \in X, j \in \overline{X}\}|)$. Show that this problem can be solved efficiently by network flow methods. Hint: Construct an appropriately defined network (graph along with capacities).

 How can the approach be adapted if also the arcs from $\overline{X}$ to X are to be counted/taken into account?

66. We are given a directed graph $G = (V, A)$ along with vertex weights $w_i \in \mathbb{R}$ for all $i \in V$. A set of vertices $S \subseteq V$ is called closed if it fulfills the following property: If $i \in S$ and $(i,j) \in A$, then also $j \in S$ holds. The task is to find a closed vertex set S with maximum weight $w(S) = \sum_{i \in S} w_i$. Prove that this problem can be transformed to a flow (cut) problem in an appropriately defined graph/network.

67. (a) Model the following problem as a minimum cut problem in an appropriately defined graph/network: We are given a set $I = \{1, \ldots, n\}$ of items along with subsets $S_i \subseteq I$, $i = 1, \ldots, m$. For each item j we are given a price $c_j > 0$ and for each subset S_i a profit $b_i > 0$ which only is obtained if all items from the S_i are available. The task is to find a subset $I' \subseteq I$ of items such that resulting revenue (= sum of the obtained profits − sum of the costs of the selected elements in I') is maximized.

 (b) Demonstrate your approach for the following specific example: $n = 6$, $m = 7$ and

Item j	1	2	3	4	5	6
Price c_j	4	5	12	6	10	5

Subset S_i	1	2	3	4	5	6	7
Elements in S_i	1,2	1,5	2,3,4	3,4	3,4,5	4,6	5,6
Profit b_i	7	4	9	3	8	6	3

68. The U.S. Census Bureau publishess tables with census data. Let $D = (d_{ij})$ with $d_{ij} \geq 0$ for all $1 \leq i \leq p$, $1 \leq j \leq q$, be a $p \times q$ table. Let r_i respectively c_j denote the sum of the entries in the i-th row respectively in the j-th column. We assume that $r_i > 0$ holds for all $1 \leq i \leq p$ and $c_j > 0$ for all $1 \leq j \leq q$. The Census Bureau publishes the sum data r_i, $1 \leq i \leq p$, and c_j, $1 \leq j \leq q$, as well as some selected entries of the table D. The office typically keeps some entries secret due to privacy reasons.

 Let Y denote the set of published entries of D. It could happen that the value of a not published entry could be recovered from the published data (including the row and column sums). This only can happy if only a single value of a not published table entry is consistent with the published data. Such a table entry will be referred to as insecure.

 Devise a polynomial algorithm to determine all insecure entries of the table D.

69. **The bottleneck transportation problem (BTP).**
 The transportation problem is a special case of the minimum cost flow problem. We are given a bipartite graph $G = (V_1 \cup V_2, E)$, a supply/demand vector b where all vertices in V_1 are sources (i.e.. $b_i > 0$ for $i \in V_1$) all vertices in V_2 are sinks (i.e.. $b_i < 0$ for $i \in V_2$) and arc costs c_{ij} for $(i,j) \in E$. There are no capacity constraints, i.e. $u_{ij} = \infty$ for all $(i,j) \in E$.

 In the bottleneck version of the transportation problem (BTP) the task is to find an integral feasible flow $x = (x_{ij})_{(i,j) \in E}$ such that the objective function $\max\{c_{ij} x_{ij} : (i,j) \in E\}$ is minimized.

(a) Consider the following decision problem version associated with the BTP: For a given real number $\lambda > 0$, decide whether there exists a feasible flow x in G with objective function value $\max\{c_{ij}x_{ij} : (i,j) \in E\} \leq \lambda$?

Devise an efficient algorithm to solve this decision problem.

(b) Use your method for (a) to develop a polynomial time algorithm to solve BTP. What is the running time of your algorithm?

70. The village Sunny has r residents $R_1, R_2, \ldots, R_r$, q clubs $C_1, C_2, \ldots, C_q$ and p political parties $P_1, P_2, \ldots, P_p$. Each resident is member in at most one party and at least one club. Each of the clubs has been asked to nominate one representative for a senate out of its set of members. The senate needs to be assembled in such a way that (i) at most u_k senate members belong to the party P_k for $k = 1, \ldots, p$ and (ii) no member of the senate represents more than one club.

(a) The question that now arises is under which conditions a feasible nomination for the senate exists at all. Prove that this question can be answered with the help of an appropriately constructed network flow problem.

(b) Consider the following specific example: Let $r = 7$, $q = 4$ and $p = 3$. Club C_1 has the members R_1 and R_2, C_2 has the members R_2, R_3 and R_4, C_3 has the members R_4 and R_5 and C_4 has the members R_4, R_5, R_6 and R_7. The residents R_1 and R_2 are members of the party P_1, R_3 and R_4 are members of party P_2 and R_5, R_6 and R_7 joined the party P_3. The capacity limits are $u_1 = 2$, $u_2 = 2$ and $u_3 = 1$.

Demonstrate your approach in (a) for this example and provide a feasible solution or argue that no such solution exists.

(c) How does the situation change when u_3 is set to 0?

71. In a regional soccer league 10 teams compete for becoming the champion. Each team plays four times against each other team. The order and the dates for the games are fixed at the beginning of the season. Suppose that (as it has been the case in Austria in former times) the winner gets 2 points, the loser gets 0 points and each team gets 1 point in case the game ends with a tie. The team with the most points at the end of the season ends up as champion. (Do not care about ties in this problem, which means that there could be several champions.)

Suppose that we want to know at some point of time throughout the season whether a certain team X can (theoretically in the best case for this team) still become champion. Of course this depends not only on the results of the team X but also on the results of all other games still to be played.

Rephrased we want to know whether there exists a scenario for the results of the games still to be played so that team X will end up with at least as many points as any of the other teams.

(a) Show that this problem can be formulated as flow problem.

(b) Does your approach also work for the case where each game has a winner which gets 1 point and a loser which gets 0 points? If so, why? If not, why not?

(c) Does your approach also work for the scoring rule now common in soccer: the winner gets 3 points, the loser gets 0 points and both teams get 1 point in case of a tie? If so, why? If not, why not?

72. Cost-effective reporting about large sports events.

At the Olympic games a group of journalists wants to cover all competitions and do not miss one. For each competition its start time, duration and location are known. Moreover, it is known how long it takes from each location to each other. The task is to cover all events by the smallest number of journalists. How can this problem be solved by network flow techniques?

73. A company can select from a given set of activities (projects, investments,...) $P = \{p_1, p_2, \ldots, p_n\}$ those that it wants to carry out. These decisions are however not independent from each other. The dependence is described by a given binary relation $A \subseteq P \times P$ where $(p, q) \in A$ means that activity p can be selected only if q is selected too. („p implies q“). The dependency graph (P, A) is assumed to be acyclic.

    Suppose that for each activity p_i we are given its monetary contribution c_i ($c_i > 0$ means that an activity brings more than it costs and if $c_i < 0$ it costs more than it brings).

    The company wants to select a feasible subset of activities from P such that the net revenue is maximized.

    Model this problem as a flow problem in an appropriately defined network.

74. Consider the following generalization of the maximum flow problem: For each arc $(i, j) \in A$ we are not only given a capacity u_{ij}, but also a travel time $\tau_{ij} \in \mathbb{N}_0$. Flow that starts at time $\tau \in \mathbb{N}_0$ in vertex i and travels along the arc (i, j), arrives in vertex j at time $\tau + \tau_{ij}$. Moreover, we are given a time horizon $T \in \mathbb{N}$.

    A maximum dynamic flow sends as many flow units from s to t within the time T as possible.

    (a) Show that the maximum dynamic flow problem can be solved as classical maximum flow problem in an appropriately extended network/graph.
    (b) What is the drawback of the approach in (a)?

75. Prove the version of the max flow min cut theorem that applies to the version with lower and upper arc capacities.

76. Given a directed graph $G = (V, A)$ along with lower capacities ℓ_{ij} and upper capacities u_{ij} for all $(i, j) \in A$, the feasible circulation problem is to identify a flow x such that $\ell_{ij} \le x_{ij} \le u_{ij}$ for all $(i, j) \in A$ and $\sum_{(i,j)\in A} x_{ij} - \sum_{(k,i)\in A} x_{ki} = 0$ for all $i \in V$ (x is called circulation).

    Prove Hoffman's circulation theorem: There exists a feasible circulation if and only if for each subset X of vertices the following condition is fulfilled

    $$\sum_{(i,j)\in(\overline{X},X)} \ell_{ij} \le \sum_{(i,j)\in(X,\overline{X})} u_{ij}.$$

    where $\overline{X} = V \setminus X$ and (Z_1, Z_2) denotes the set of arcs in A with $i \in Z_1$ and $j \in Z_2$.

77. Determine a maximum flow in the network displayed in Figure 11. The numbers ℓ_{ij}, u_{ij} attached to the arcs are the lower and upper arc capacities, respectively.

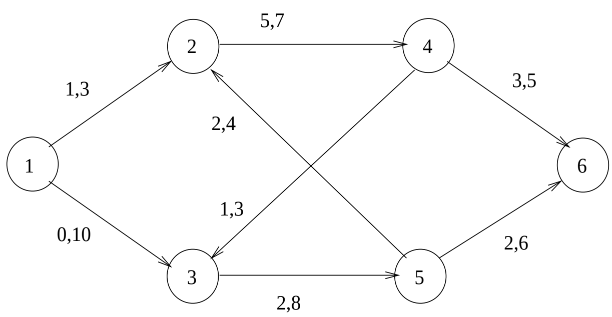

Figure 11

78. (a) In which of the networks displayed in Figure 12 (consider all arcs as arcs which are directed from the vertex with the lower number of the vertex with the higher number) does there exist a feasible flow from the source $s = 1$ to the sink $t = 6$ with respect to the lower and upper arc capacities ℓ_{ij} and u_{ij} attached to the arc (i, j).

(b) When a feasible flow exists, provide a feasible flow with maximum value and a feasible flow with minimum value. If no feasible flow exists, provide a cut that violates Hoffman's condition.

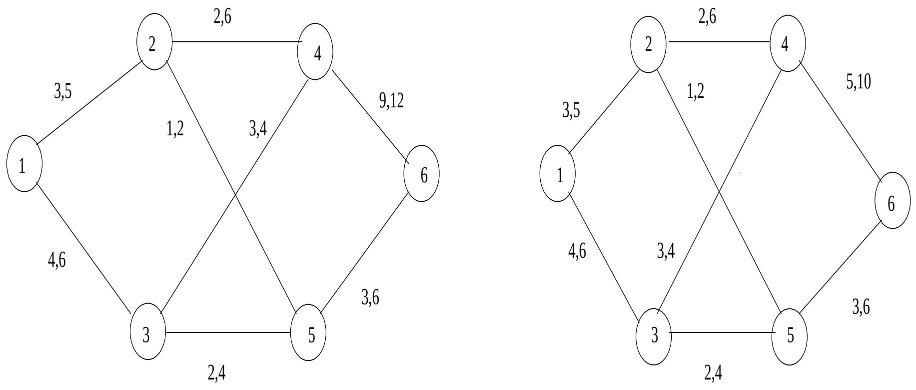

Figure 12

79. Given are an instance of the maximum flow problem and a maximum flow for this instance.

    (a) Suppose that the capacity of a single arc (i, j) increases by $k > 0$. Show how one can find a new maximum flow within $O(mk)$ time (m denotes the number of arcs as usual).

    (b) As in (a) but for the case where the capacity of the arc (i, j) decreases by $k > 0$ with $k < u_{ij}$.

80. Consider the minimum cut problem. Let $(X, \overline{X})$ and $(Y, \overline{Y})$ be two s-t cuts (cuts for short) in the graph G. Prove that the cut capacity function is *submodular*, i.e. that the following holds

$$u(X, \overline{X}) + u(Y, \overline{Y}) \geq u(X \cup Y, \overline{X \cup Y}) + u(X \cap Y, \overline{X \cap Y}) . \tag{1}$$

    Use property (1) to prove the following property of minimum cuts: The union and the intersection of two minimum cuts $(X \cup Y, \overline{X \cup Y})$ and $(X \cap Y, \overline{X \cap Y})$ are minimum cuts, too.

81. We are given an instance of the maximum flow problem with integral capacities and a non-integral flow maximum flow x^*. Devise an efficient algorithm to transform the flow x into an integral maximum flow. Analyse the running time of your algorithm.

82. We are given a directed graph G along with a source s, a sink t and lower and upper nonnegative arc capacities ℓ_{ij} and u_{ij} respectively. Suppose that at least one of the lower arc capacities ℓ_{ij} is strictly larger than 0. Then the zero flow is not feasible. It thus makes sense to ask for a feasible flow with minimum flow value (= "minimum flow value problem").

    (a) Find a notion that corresponds in this problem to the concept of an augmenting path in the maximum flow problem. Develop an optimality criterion for the minimum flow value problem based on this concept.

    (b) Devise an algorithm to solve the minimum flow value problem and demonstrate your algorithm with the help of a small example.

    (c) Prove the following: If there exists a feasible flow then the minimum flow value of a feasible flow equals the maximum of $\ell(X, \overline{X}) - u(\overline{X}, X)$ over all s-t cuts $(X, \overline{X})$.

83. Let $G = (V, E)$ be a simple (no loops, no parallel edges) undirected graph with at least two vertices. Suppose that all vertex degrees are $\geq k \in \mathbb{N}$. Prove that there exist two vertices s and t such that at least k edge disjoint $s - t$ paths exists. What if there is exactly one vertex with degree less than k? (Hint: Gomory-Hu tree)

84. Prove the flow decomposition theorem presented in the lecture.