49. Apply the Bellman Ford Moore algorithm to determine a shortest path tree with start/root vertex $s = 1$ for the graph displayed in Figure 5.

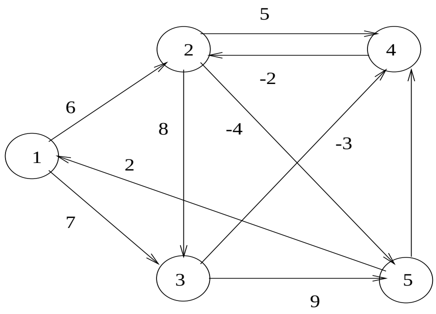

Figure 5

50. Determine shortest paths between all vertex pairs for the graphs from Figure 6.

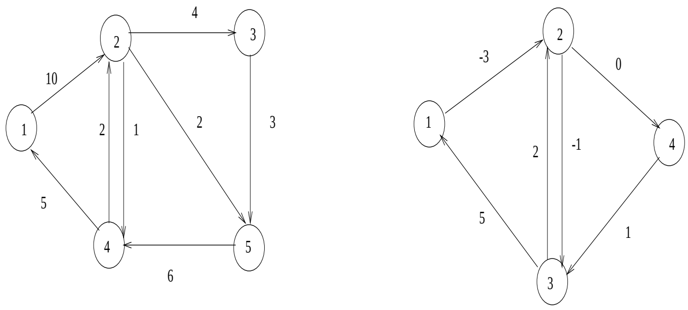

Figure 6

51. Devise an algorithm for finding a shortest path tree in an acyclic directed graph (does not contain a directed cycle) that is faster than the known shortest path algorithms for general graphs.

Demonstrate your algorithm for the graph from Figure 7 (let $s = 1$).

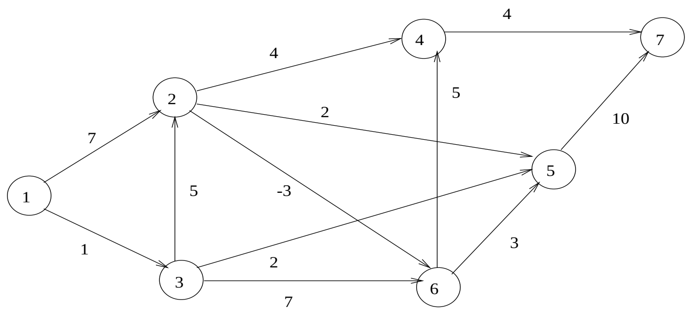

Figure 7

52. Mr Clever suggest the following approach for the shortest path problem with arbitary real arc weight w_{ij}: Let $w_{\min} = \min\{w_{ij} : (i,j) \in A\}$. If $w_{\min} < 0$, solve the shortest path problem with the nonnegative arc weights $w'_{ij} = w_{ij} + |w_{\min}|$ with Dijkstra's algorithm. Is this approach correct? Either provide a proof or a counterexample.

53. Let a directed graph $G = (V, A)$ arc weights w_{ij}, $(i,j) \in A$, and two distinguished vertices s and t be given. Let P^* be a shortest path from s to t and denote by $\mathcal{P}_{st}$ the set of all directed paths from s

to t. Devise an efficient algorithm for solving the following problem (second-shortest path problem): Find a path P' from s to t such that

$$w(P') \leq w(P) \qquad \text{für alle } P \in \mathcal{P}_{st} \setminus \{P^*\},$$

i.e., an st path which is a shortest st path with the exception of P ($w(P^*) = w(P')$ can happen).

54. Given a directed graph $G = (V, A)$, arc weights w_{ij}, $(i, j) \in A$ and a start vertex $s \in V$ and a target vertex $t \in V$. The task is to the find an st path which maximizes the bottleneck weight $w(P) = \min_{(i,j)\in P} w_{ij}$.
    (a) Modify Dijkstra's algorithm to apply to this situation.
    (b) Suppose that a list of the arcs sorted with respect to nonincreasing weight is available. Devise an alternative algorithm which makes use of this list and analyse it.

55. Consider the following system of difference inequalities: Find a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ which fulfills the given inequalities

$$x_{j_k} - x_{i_k} \leq b_k \qquad \text{for } k = 1, \dots, m$$

where $i_k, j_k \in \{1, \dots, n\}$ and $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ is a given vector of real numbers.

Prove that such a vector x exists if and only if there does not exist a negative cycle in the directed graph we constructed from the inequality system in the lecture. Also explain how a solution vector x can be found if the system is feasible.

56. Find a solution to the following inequality systems or prove that no solution exists:

$$(a) \quad \begin{array}{rcl} x_1 - x_2 & \leq & 4 \\ x_1 - x_5 & \leq & 2 \\ x_2 - x_4 & \leq & -6 \\ x_3 - x_2 & \leq & 1 \\ x_4 - x_1 & \leq & 3 \\ x_4 - x_3 & \leq & 5 \\ x_4 - x_5 & \leq & 10 \\ x_5 - x_3 & \leq & -4 \\ x_5 - x_4 & \leq & -8 \end{array} \qquad (b) \quad \begin{array}{rcl} x_1 - x_2 & \leq & 1 \\ x_1 - x_4 & \leq & -4 \\ x_2 - x_3 & \leq & 2 \\ x_2 - x_5 & \leq & 7 \\ x_2 - x_6 & \leq & 5 \\ x_3 - x_6 & \leq & 10 \\ x_4 - x_2 & \leq & 2 \\ x_5 - x_1 & \leq & -1 \\ x_5 - x_4 & \leq & 3 \\ x_6 - x_3 & \leq & -8 \end{array}$$

57. Given is a directed graph $G = (V, A)$, a start vertex $s \in V$ and a target vertex $t \in V$. Further, a weight $w_{ij} \in \mathbb{R}$ is attached to each arc (i, j) and a potential $\pi_i \in \mathbb{R}$ to each vertex $i \in V$. The potential difference $\tau_{ij} = \pi_i - \pi_j$ is referred to as *tension* of the arc (i, j) with respect to the potential vector π. Let $\mathcal{P}$ denote the set of all directed paths from s to t.

Consider the following problem (maximum tension problem):

$$\begin{array}{lll} (MT) \quad \max & \sum\limits_{(i,j)\in P} \tau_{ij} & \\ & & \\ s.t. & P \in \mathcal{P} & \\ & \tau_{ij} = \pi_i - \pi_j & \text{for all } (i,j) \in P \\ & \tau_{ij} \leq w_{ij} & \text{for all } (i,j) \in P \\ & \pi_i \in \mathbb{R} & \text{for all } i \in V. \end{array}$$

If P^* is a path for which the maximum in (MT) is achieved, the value $\sum_{(i,j)\in P^*} \tau_{ij}$ is referred to as maximum tension between s and t.

(a) Prove that under the assumption that a *st* shortest path exists with respect to the arc weights w_{ij}, the length of a shortest *st* path equals the maximum tension between s and t.

(b) (For those familiar with linear programming duality): Interpret the result from (a) as a duality result.

(c) Provide an example that demonstrates that (a) is correct for that example.

58. Suppose that the arcs weights w_{ij} are integers from the set $\{0, \ldots, C\}$ where C is a constant. Show that Dijkstra'a algorithm for this special case can be implemented to run in linear time. (Hint: Use an array indexed by 0,…, nC.)

59. Prove that the Floyd Warshall algorithm also works for arbitrary real weights.

60. We consider the all pairs shortest path problem. Let $\tilde{d}_{ij}^{(k)}$ denote the length of a shortest path from i to j with at most k arcs.

    Provide a dynamic programming recursion for computing $\tilde{D}^{(k+1)}$ from $\tilde{D}^{(k)}$. This leads to an $O(n^4)$ algorithm. Also explain how to obtain the shortest paths. For extra credit: Improve the running time.

61. In the most reliable path problem we are given a directed graph $G = (V, A)$, arc reliabilities $r_{ij} \in [0, 1]$ and two vertices $s, t \in V$. The task is to find an *st* path which maximizes $\Pi_{(i,j)\in P} r_{ij}$.

    (a) Modify Dijkstra's algorithm to solve this problem directly.

    (b) Transform the problem into a classical shortest path problem.

62. Decide for each of the following statements whether they are true or false (provide a proof or a counterexample).

    (a) If there is a unique minimum cut, there is a unique maximum flow.

    (b) If there is a unique maximum flow, there is a unique minimum cut.

    (c) If all capacities in an instance of the maximum flow problem are divisible by 3, there always exists a maximum flow x such that all x_{ij} are divisible by 3.

    (d) If all capacities in an instance of the maximum flow problem are odd, there always exists a maximum flow x such that all x_{ij} are odd.

    (e) If the graph $G = (V, A)$ contains an arc (i, j) with a capacity u_{ij} such that $u_{ij} > u_{k\ell}$ for all arcs $(k, \ell) \in A \setminus \{(i, j)\}$, then there does not exist a minimum cut $(X, \overline{X})$ which contains this arc, i.e. for which $i \in X$ and $j \in \overline{X}$.

    (f) Multiplying the capacities u_{ij} in a minimum cut problem by a constant $k > 0$, does not change the set of minimum cuts.

    (g) Adding a constant $k > 0$ to the capacities u_{ij} in a maximum flow problem, results in a maximum flow value that increases by a multiple of k (with respect to the maximum flow value of the original instance).

    (h) Given a flow x. It can be decided in $O(m + n)$ time whether x is a unique maximum flow.

63. The members of a new savings club "Penny Pinchers" organize a Christmas party in the pub "Golden Fox". To encourage that the members get to know each other, no two persons from the same family should get assigned to the same table. There are q tables available and it is known that table j offers seats for b_j, $j = 1, \ldots, q$ persons. Moreover, it is known that a_i members of family i, $i = 1, \ldots, p$ belong to the club where p denotes the overall number of represented families. The task is to find a feasible assignment of the members to the tables.

    (a) Formulate this problem as a flow problem on an appropriately defined graph/network.

(b) Does there always exist a feasible assignment? (Explain your answer.)

64. Determine a maximum flow and a minimum cut (with an approach of your choice) for the networks displayed in Figures 8 and 9 (set $s = 1$ and $t = 8$).

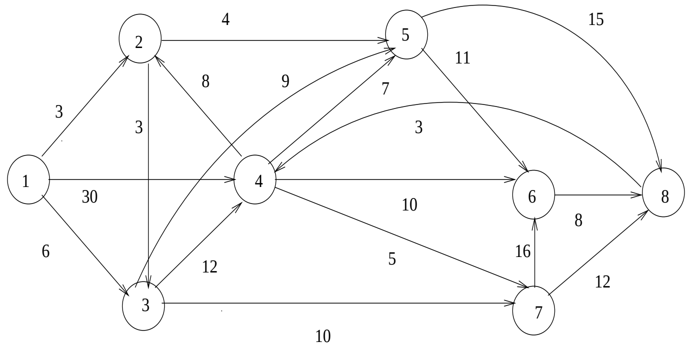

Figure 8

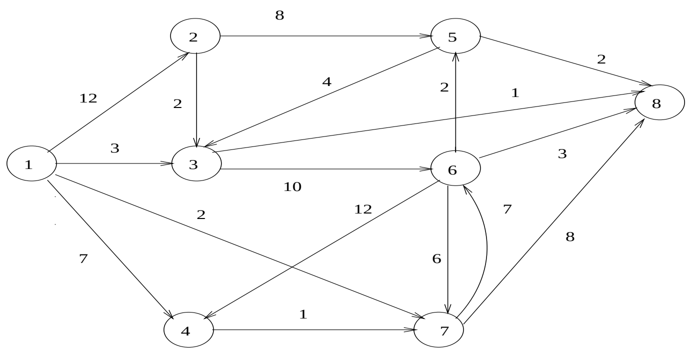

Figure 9

65. Consider the graph from Figure 10. All arcs with exception of the arcs $(3, 6)$ with capacity $u_{36} = 1$ and $(1, 2)$ and $(4, 6)$ with capacity $u_{12} = u_{46} = \sigma = (-1 + \sqrt{5})/2$ have a capacity $u_{ij} > 1$.

(a) Show that by a bad choice of the augmenting paths in Ford and Fulkerson's algorithm one ends up with a sequence of flows with the property that the sequence of the values of these flows converge to a value which is (arbitrarily) smaller than the maximum flow value v_{max}.
Hint 1: Note that $\sigma^{k+2} = \sigma^{k+1} - \sigma^k$, $k \geq 0$.
Hint 2: Send flows repeated along the following six paths:
$P_1 = (s, 1, 2, 3, 6, t)$, $P_2 = (s, 2, 1, 3, 6, 5, t)$, $P_3 = (s, 1, 2, 4, 6, t)$, $P_4 = (s, 2, 1, 4, 6, 3, t)$, $P_5 = (s, 1, 2, 5, 6, t)$, $P_6 = (s, 2, 1, 5, 6, 4, t)$.

(b) Solve the instance from (a) by choosing augmenting paths with minimum number of arcs.

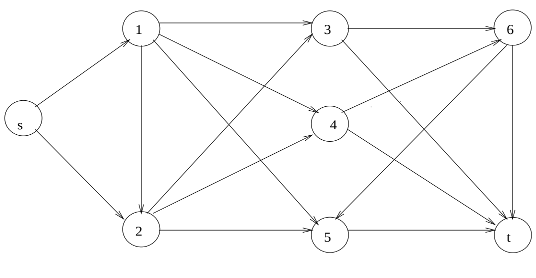

Figure 10