Ordinary Generating Functions

$$A(z) = \sum_{n\geq 0} a_n z^n \qquad a_0, a_1, a_2, \ldots, a_n, \ldots,$$

$$B(z) = \sum_{n\geq 0} b_n z^n \qquad b_0, b_1, b_2, \ldots, b_n, \ldots,$$

right shift

$$zA(z) = \sum_{n\geq 1} a_{n-1} z^n \qquad 0, a_0, a_1, a_2, \ldots, a_{n-1}, \ldots,$$

left shift

$$\frac{A(z) - a_0}{z} = \sum_{n\geq 0} a_{n+1} z^n \qquad a_1, a_2, \ldots, a_{n+1}, \ldots,$$

index multiply (differentiation)

$$A'(z) = \sum_{n\geq 0} (n+1) a_{n+1} z^n \qquad a_1, 2a_2, \ldots, (n+1)a_{n+1}, \ldots,$$

index divide (integration)

$$\int_0^z A(t)\,dt = \sum_{n\geq 1} \frac{a_{n-1}}{n} z^n \qquad 0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots, \frac{a_{n-1}}{n}, \ldots,$$

scaling

$$A(\lambda z) = \sum_{n\geq 0} \lambda^n a_n z^n \qquad a_0, \lambda a_1, \lambda^2 a_2, \ldots, \lambda^n a_n, \ldots,$$

addition

$$A(z) + B(z) = \sum_{n\geq 0} (a_n + b_n) z^n \qquad a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n, \ldots,$$

difference

$$(1-z)A(z) = a_0 + \sum_{n\geq 1} (a_n - a_{n-1}) z^n \qquad a_0, a_1 - a_0, \ldots, a_n - a_{n-1}, \ldots,$$

convolution

$$A(z)B(z) = \sum_{n\geq 0} \left(\sum_{0\leq k\leq n} a_k b_{n-k} \right) z^n \qquad a_0 b_0, a_1 b_0 + a_0 b_1, \ldots, \sum_{0\leq k\leq n} a_k b_{n-k}, \ldots,$$

partial sum

$$\frac{A(z)}{1-z} = \sum_{n\geq 0} \left(\sum_{0\leq k\leq n} a_k \right) z^n \qquad a_1, a_1 + a_2, \ldots, \sum_{0\leq k\leq n} a_k, \ldots,$$

Exponential Generating Functions

$$A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!} \qquad a_0, a_1, a_2, \ldots, a_n, \ldots,$$

$$B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!} \qquad b_0, b_1, b_2, \ldots, b_n, \ldots,$$

right shift (integration)

$$\int_0^z A(t)\,dt = \sum_{n\geq 1} a_{n-1} \frac{z^n}{n!} \qquad 0, a_0, a_1, \ldots, a_{n-1}, \ldots,$$

left shift (differentiation)

$$A'(z) = \sum_{n\geq 0} a_{n+1} \frac{z^n}{n!} \qquad a_1, a_2, a_3, \ldots, a_{n+1}, \ldots,$$

index multiply

$$zA(z) = \sum_{n\geq 0} n a_{n-1} \frac{z^n}{n!} \qquad 0, a_0, 2a_1, 3a_2, \ldots, n a_{n-1}, \ldots,$$

index divide

$$\frac{A(z) - A(0)}{z} = \sum_{n\geq 1} \frac{a_{n+1}}{n+1} \frac{z^n}{n!} \qquad a_1, \frac{a_2}{2}, \frac{a_3}{3}, \ldots, \frac{a_{n+1}}{n+1}, \ldots,$$

addition

$$A(z) + B(z) = \sum_{n\geq 0} (a_n + b_n) \frac{z^n}{n!} \qquad a_0 + b_0, \ldots, a_n + b_n, \ldots,$$

difference

$$A'(z) - A(z) = \sum_{n\geq 0} (a_{n+1} - a_n) \frac{z^n}{n!} \qquad a_1 - a_0, a_2 - a_1, \ldots, a_{n+1} - a_n, \ldots,$$

binomial convolution

$$A(z)B(z) = \sum_{n\geq 0} \left(\sum_{0\leq k\leq n} \binom{n}{k} a_k b_{n-k} \right) \frac{z^n}{n!} \qquad a_0 b_0, a_1 b_0 + a_0 b_1, \ldots, \sum_{0\leq k\leq n} \binom{n}{k} a_k b_{n-k}, \ldots,$$

binomial sum

$$e^z A(z) = \sum_{n\geq 0} \left(\sum_{0\leq k\leq n} \binom{n}{k} a_k \right) \frac{z^n}{n!} \qquad a_0, a_0 + a_1, \ldots, \sum_{0\leq k\leq n} \binom{n}{k} a_k, \ldots,$$