Tutorium Mathe II M WM VT SS 2011

13.5.2011

Lösung des 2. Beispiels

$$I := \iint_M \frac{4(x_1+x_2)}{(x_1^2+x_2^2)^3}\, dx_1\, dx_2$$

$$M = \left\{ x \in \mathbb{R}^2 : \; 0 < \frac{x_2}{x_1^2+x_2^2} < 1 - \frac{x_1}{x_1^2+x_2^2} < \frac{1}{2} \right\}$$

Substitution: $u_1 = \dfrac{x_1}{x_1^2+x_2^2}$ $\quad u_2 = \dfrac{x_2}{x_1^2+x_2^2}$

$$\frac{\partial u_1}{\partial x_1} = \frac{x_1^2+x_2^2-2x_1\cdot x_1}{(x_1^2+x_2^2)^2} = \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}$$

$$\frac{\partial u_1}{\partial x_2} = -\frac{2x_1x_2}{(x_1^2+x_2^2)^2}$$

Analog: $\dfrac{\partial u_2}{\partial x_1} = \dfrac{-2x_1x_2}{(x_1^2+x_2^2)^2}$ $\quad \dfrac{\partial u_2}{\partial x_2} = \dfrac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2}$

$$\left| J_{(x_1x_2)}(u_1u_2) \right| = \begin{vmatrix} \dfrac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2} & -\dfrac{2x_1x_2}{(x_1^2+x_2^2)^2} \\[2ex] \dfrac{-2x_1x_2}{(x_1^2+x_2^2)^2} & \dfrac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2} \end{vmatrix}$$

$$= -\frac{(x_1^2-x_2^2)^2}{(x_1^2+x_2^2)^4} - \frac{(2x_1x_2)^2}{(x_1^2+x_2^2)^4} = -\frac{(x_1^2+x_2^2)^2}{(x_1^2+x_2^2)^4}$$

$$\left|J_{(x_1x_2)}(u_1u_2)\right| = -\frac{1}{(x_1^2+x_2^2)^2}$$

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$$du_1\,du_2 = \left|\left|J_{x_1x_2}(u_1u_2)\right|\right|\,dx_1\,dx_2$$

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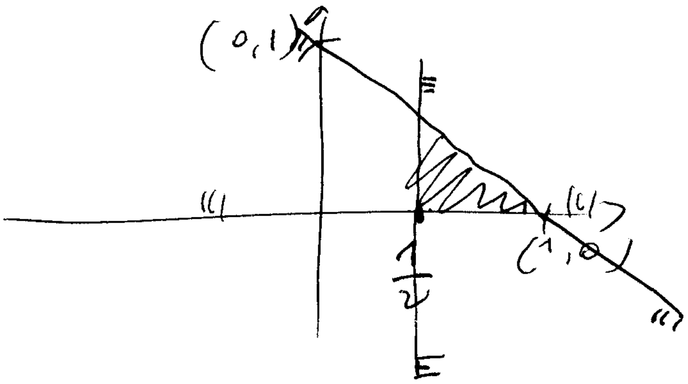
$$du_1\,du_2 = \frac{1}{(x_1^2+x_2^2)}\,dx_1\,dx_2$$

Transformierter Bereich M':

$$M' = \left\{(u_1,u_2)\in\mathbb{R}^2 : 0 < u_2 < 1-u_1 < \tfrac{1}{2}\right\}$$



Beobachtung

$$u_1+u_2 = \frac{x_1+x_2}{x_1^2+x_2^2}$$

$$I = \iint_{M} \frac{4(x_1+x_2)}{(x_1^2+x_2^2)^3}\,dx_1\,dx_2 = \iint_{M'} 4(u_1+u_2)\underbrace{\frac{dx_1\,dx_2}{(x_1^2+x_2^2)^2}}_{du_1\,du_2}$$

$$= \int_{u_1=\frac{1}{2}}^{1}\int_{u_2=0}^{1-u_1} 4(u_1+u_2)\,du_2\,du_1 =$$

$$= \int_{\frac{1}{2}}^{1} \left(4u_1u_2 + 2u_2^2\right)\Big|_0^{1-u_1} du_1 =$$

$$= \int_{1/2}^{1} \left[4u_1(1-u_1) + 2(1-u_1)^2\right] du_1 = \int_{\frac{1}{2}}^{1} (2u_1^2-2)\,du_1 = \left(\frac{2u_1^3}{3} - 2u_1\right)\Big|_{1/2}^{1} = -\frac{5}{12}$$

$$\boxed{I = -\frac{5}{12}}$$