Aufgabe 26.11 ●● Mehrfaches Ableiten der ersten Kurve liefert

$$\boldsymbol{\alpha}(t) = \begin{pmatrix}\cosh t\\ \sinh t\\ t\end{pmatrix}, \qquad \dot{\boldsymbol{\alpha}}(t) = \begin{pmatrix}\sinh t\\ \cosh t\\ 1\end{pmatrix},$$
$$\ddot{\boldsymbol{\alpha}}(t) = \begin{pmatrix}\cosh t\\ \sinh t\\ 0\end{pmatrix}, \qquad \dddot{\boldsymbol{\alpha}}(t) = \begin{pmatrix}\sinh t\\ \cosh t\\ 0\end{pmatrix}.$$

Damit erhalten wir für Krümmung und Torsion:

$$\begin{aligned}
\kappa &= \frac{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|}{\|\dot{\boldsymbol{\alpha}}\|^3} \\
&= \frac{1}{\left(\sinh^2 t + \cosh^2 t + 1\right)^{3/2}} \left\| \begin{pmatrix}-\sinh t\\ \cosh t\\ -1\end{pmatrix} \right\| \\
&= \frac{1}{2\cosh^2 t} \\
\tau &= \frac{\det((\dot{\boldsymbol{\alpha}},\ \ddot{\boldsymbol{\alpha}},\ \dddot{\boldsymbol{\alpha}}))}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|^2} \\
&= \frac{1}{2\cosh^2 t} \begin{vmatrix}\sinh t & \cosh t & \sinh t\\ \cosh t & \sinh t & \cosh t\\ 1 & 0 & 0\end{vmatrix} \\
&= \frac{1}{2\cosh^2 t}
\end{aligned}$$

Das begleitende Dreibein ergibt sich zu:

$$\begin{aligned}
\widehat{\boldsymbol{t}}(t) &= \frac{1}{\|\dot{\boldsymbol{\alpha}}(t)\|}\,\dot{\boldsymbol{\alpha}}(t) = \frac{1}{\sqrt{2}\cosh t}\begin{pmatrix}\sinh t\\ \cosh t\\ 1\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}\tanh t\\ 1\\ 1/\cosh t\end{pmatrix} \\
\widehat{\boldsymbol{h}} &= \frac{1}{\|\dot{\widehat{\boldsymbol{t}}}(t)\|}\,\dot{\widehat{\boldsymbol{t}}}(t) = \frac{1}{\|\dot{\widehat{\boldsymbol{t}}}(t)\|}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1-\tanh^2 t\\ 0\\ -\sinh t/\cosh^2 t\end{pmatrix} \\
&= \cosh t \begin{pmatrix}1+\tanh^2 t\\ 0\\ -\sinh t/\cosh^2 t\end{pmatrix} \\
&= \begin{pmatrix}\cosh t + \sinh t\ \tanh t\\ 0\\ -\tanh t\end{pmatrix} \\
\widehat{\boldsymbol{b}} &= \widehat{\boldsymbol{t}} \times \widehat{\boldsymbol{h}} = \frac{1}{\sqrt{2}}\begin{pmatrix}\tanh t\\ 1-2\tanh^2 t\\ \sinh t\ \tanh t - \cosh t\end{pmatrix}
\end{aligned}$$

Für die Bogenlänge erhalten wir:

$$\begin{aligned}
s &= \int_0^t \sqrt{\cosh^2\tau + \sinh^2\tau + 1}\,\mathrm{d}\tau \\
&= \sqrt{2}\int_0^t \cosh\tau\,\mathrm{d}\tau = \sqrt{2}\sinh t
\end{aligned}$$