Tutorium am 29.6.2011 Mathe II MWM VT 1. Beispiel

Notiztitel 24.06.2011

$$x_1' = -2x_1 - 4x_2$$

$$x_2' = -x_1 + x_2 - 10e^{-3t} + 5\cos 2t + \sin 2t$$

$$x_1(0) = 3 \qquad x_2(0) = \frac{5}{4}$$

1 Lösung des homog. Systems

$$x_1' = -2x_1 - 4x_2$$

$$x_2' = -x_1 + x_2$$

$$A = \begin{pmatrix} -2 & -4 \\ -1 & 1 \end{pmatrix}$$

Koeff.matrix

Eigenwerte von A

$$\left|\begin{pmatrix} -2-\lambda & -4 \\ -1 & 1-\lambda \end{pmatrix}\right| = 0 \quad \Leftrightarrow \quad (2+\lambda)(\lambda-1) - 4 = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = -3$$

Fundamentalsystem $\underline{e^{2t}, \; e^{-3t}}$

<u>Eigenvektor zu $\lambda_1 = 2$</u>

$$(A - 2 \cdot I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} -4 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$-v_1 - v_2 = 0 \implies v_1 = -v_2 \implies \underline{\underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}} \text{ Eigenvektor zu } \lambda_1 = 2$$

Eigenvektor zu $\lambda_2 = -3$

$$(A + 3I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \implies \underline{\underline{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}}$$

Allg. Lösung des homog. Systems

$$\underbrace{\begin{pmatrix} x_{1h} \\ x_{2h} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-3t}}$$

Suche partikuläre Lösungen

$$b = \begin{pmatrix} 0 \\ -10\,e^{-3t} + 5\cos 2t + \sin 2t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ -10\,e^{-3t} \end{pmatrix}}_{b_1} + \underbrace{\begin{pmatrix} 0 \\ 5\cos 2t + \sin 2t \end{pmatrix}}_{b_2}$$

Sei $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = x^*$ eine partikuläre Lösung zu b_1.

Hauptachsentransformation

$$R = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix} \qquad R^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \qquad |R| = 5$$

EV λ_1 EV λ_2

$$\underline{x^* = R z^*} \Rightarrow \quad x^{*\prime} = A x^* + b_1$$

$$R z^{*\prime} = A R z^* + b_1$$

$$z^{*\prime} = \underbrace{R^{-1} A R}_{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} z^* + R^{-1} b_1$$

$$z^{*\prime} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} z_1^* \\ z_2^* \end{pmatrix} + R^{-1} b_1$$

$$R^{-1} b_1 = \frac{1}{5}\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -10 e^{-3t} \end{pmatrix} = \begin{pmatrix} 8 e^{-3t} \\ -2 e^{-3t} \end{pmatrix}$$

$$(1)\quad z_1^{*'} = 2 z_1^* + 8 e^{-3t} \longrightarrow z_1^{*'} = 2 z_1^*$$

$$(2)\quad z_2^{*'} = -3 z_2^* - 2 e^{-3t}$$

$$z_2^{*'} = -3 z_2^*$$

$$\frac{dz_2^*}{dt} = -3 z_2^* \Rightarrow \frac{dz_2^*}{z_2^*} = -3\,dt$$

$$\ln|z_2^*| = -3t + C \Rightarrow z_2^* = Ce^{-3t}$$

$z_2^* = c(t)\, e^{-3t} \to$ setze in (2) ein

$$\frac{dz_1^*}{dt} = 2 z_1^* \Rightarrow \frac{dz_1^*}{z_1^*} = 2\,dt$$

$$\int \dots = \int 2\,dt \Rightarrow \ln|z_1^*| = 2t + C$$

$$z_1^* = Ce^{2t}$$

$z_1^* = c(t)\, e^{2t}$ setze in (1) ein

$$c'(t)\,e^{2t} + \cancel{2c(t)\,e^{2t}} = \cancel{2c(t)\,e^{2t}} + 8e^{-3t}$$

$$c'(t) = 8e^{-5t} \Rightarrow c(t) = \int 8e^{-5t}\,dt$$

$$c(t) = -8/5\, e^{-5t}$$

$$c'(t)\, e^{-3t} - \cancel{3c(t)\, e^{-3t}} = \cancel{-3c(t)\, e^{-3t}} - 2e^{-3t}$$

$$c'(t) = -2 \Rightarrow c(t) = -2t$$

$$z_2^* = -2t \cdot e^{-3t}$$

$$z_1^* = -\frac{8}{5} e^{-5t} e^{2t} = -\frac{8}{5} e^{-3t}$$

$$\begin{pmatrix} z_1^* \\ z_2^* \end{pmatrix} = \begin{pmatrix} -8/5\, e^{-3t} \\ -2t\, e^{-3t} \end{pmatrix}$$

Rücksubstitution

$$X^* = R \begin{pmatrix} -8/5 \\ -2t \end{pmatrix} e^{-3t} = \begin{pmatrix} -8/5\, e^{-3t} - 8t\, e^{-3t} \\ \frac{8}{5} e^{-3t} - 2t\, e^{-3t} \end{pmatrix}$$

Partikuläre Lösung zu b_2:

Ansatz $$x^{*(2)} = \begin{pmatrix} A\cos 2t + B\sin 2t \\ C\cos 2t + D\sin 2t \end{pmatrix}$$

Setze $x^{*(2)}$ in $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b_2$

$$\begin{pmatrix} -2A\sin 2t + 2B\cos 2t \\ -2C\sin 2t + 2D\cos 2t \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} + \begin{pmatrix} 0 \\ 5\cos 2t + \sin 2t \end{pmatrix}$$

$$-2A\sin 2t + 2B\cos 2t = (-A-4C)\cos 2t + (2B-4D)\sin 2t$$

$$-2C\sin 2t + 2D\cos 2t = (A+C)\cos 2t + (-B+D)\sin 2t + 5\cos 2t + \sin 2t$$

$$\left.\begin{aligned} -2A &= -2B - 4D \\ 2B &= -2A - 4C \\ -2C &= -B + D + 1 \\ 2D &= -A + C + 5 \end{aligned}\right\} \Rightarrow M \cdot \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} \\ \end{pmatrix}$$

$$\Rightarrow D = 1 \quad A = 2 \quad C = -1 \quad B = 0$$

$$X^{*(2)} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix} = \begin{pmatrix} 2\cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix}$$

Allg. Lösung des inhom. Systems

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-3t}}_{\begin{pmatrix} x_{1h} \\ x_{2h} \end{pmatrix}} + \begin{pmatrix} -8/5\, e^{-3t} - 8t e^{-3t} \\ \frac{8}{5} e^{-3t} - 2t e^{-3t} \end{pmatrix} + \begin{pmatrix} 2\cos 2t \\ -2\cos 2t + \sin 2t \end{pmatrix}$$

Lösung des AWP:

$x_1(0) = 1 \qquad x_2(0) = \frac{5}{4}$

$$\left.\begin{aligned} x_1(0) &= c_1 + 4c_2 - \frac{8}{5} + 2 = 1 \\ x_2(0) &= -c_1 + c_2 + 8/5 - 2 = 5/4 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= 9/20 \\ c_1 &= -6/5 \end{aligned}$$

Alternativer Lösungsweg : Transf. zu lin. DGL 2. Ordnung

$$x_1' = -2x_1 - 4x_2$$

$$x_2' = -x_1 + x_2 - 10e^{-3t} + 5\cos 2t + \sin 2t$$

$$x_2 = -\frac{1}{4}(x_1' + x_1) \Rightarrow x_2' = -\frac{1}{4}(x_1'' + x_1')$$

Setze in 2. Gleichung ein:

$$-\frac{1}{4}(x_1'' + x_1') = -x_1 - \frac{1}{4}(x_1' + x_1) + \dots \text{Störterm}$$