Tut. Mathe II 10.6 2011. Lösung Bsp 1

$$x(t) = \begin{pmatrix} 2t \\ t\sin t \\ t\cos t \end{pmatrix}$$

Länge zwischen $P_1 = (\pi, \frac{\pi}{2}, 0)$

$P_2 = (-\pi, \frac{\pi}{2}, 0)$

$$L = \int_{P_1}^{P_2} dl = \int_{P_1}^{P_2} \|\dot{x}\| \, dt$$

$$\dot{x} = \begin{pmatrix} 2 \\ \sin t + t\cos t \\ \cos t - t\sin t \end{pmatrix}$$

$$\|\dot{x}\|^2 = 4 + \sin^2 t + \underline{t^2\cos^2 t} + \cancel{2t\sin t\cos t}$$
$$+ \cos^2 t + \underline{t^2\sin^2 t} - \cancel{2t\sin t\cos t}$$

$$\|\dot{x}\|^2 = 5 + A^2$$

$P_1 \longrightarrow t_1 = \frac{\pi}{2}$

$P_2 \rightarrow t_2 = -\frac{\pi}{2}$

$$L = \int_{\frac{\pi}{2}}^{-\pi/2} \sqrt{5+t^2}\,dt = -\int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{5+t^2}\,dt = -2\int_{0}^{\pi/2} \sqrt{5+t^2}\,dt =$$

$$= -2\int_{0}^{\frac{\pi}{2}} \sqrt{5}\,\sqrt{1+\frac{t^2}{5}}\,dt = -2\sqrt{5}\int_{0}^{\frac{\pi}{2}} \sqrt{1+\frac{t^2}{5}}\,dt$$

$$\sqrt{\frac{t^2}{5}} = \sinh u \Rightarrow u = \operatorname{asinh}\sqrt{\frac{t^2}{5}}$$

$$\hookrightarrow 1+\frac{t^2}{5} = 1+\sinh^2 u = \cosh^2 u$$

$$\hookrightarrow t = \sqrt{5}\sinh u \Rightarrow dt = \sqrt{5}\cosh u\,du$$

$$0 \to \operatorname{asinh}\left(\frac{0}{\sqrt{5}}\right) = 0$$

$$\frac{\pi}{2} \to \operatorname{asinh}\left(\frac{\pi/2}{\sqrt{5}}\right) = a$$

$$L = -2\sqrt{5}\int_0^a \sqrt{5}\cosh^2 u \, du = -10 \int_0^a \cosh^2 u \, du$$

$$\int \cosh^2 u = \int \underbrace{\cosh u}_{u'} \; \underbrace{\cosh u}_{v} \, du = \dots$$

$$\Rightarrow \int \cosh^2 u = \frac{\sinh u \cosh u + u}{2}$$

$$L = -10 \, \frac{\sinh a \cosh a + a}{2} = -7.5659 \; //$$

Krümmung $\quad k = \dfrac{\sqrt{\|\dot{x}\|^2 \|\ddot{x}\|^2 - \langle \dot{x}, \ddot{x} \rangle^2}}{\|\dot{x}\|^3}$ $\quad P(2\pi, 0, \pi)$

$\Downarrow$

$t = \pi$

$$\ddot{X} = \begin{pmatrix} 0 \\ \cos t + \cos t - t \sin t \\ -\sin t - \sin t - t \cos t \end{pmatrix}$$

$$\dot{x}(\pi) = \begin{pmatrix} 2 \\ -\pi \\ -1 \end{pmatrix} \quad \|\dot{x}\|^2 = 5 + \pi^2$$

$$\ddot{x}(\pi) = \begin{pmatrix} 0 \\ -2 \\ \pi \end{pmatrix} \quad \|\ddot{x}\|^2 = 4 + \pi^2$$

$$\langle \dot{x}(\pi), \ddot{x}(\pi) \rangle = +2\pi - \pi = \pi$$

$$k = \frac{\sqrt{(5+\pi^2)(4+\pi^2) - \pi^2}}{(5+\pi^2)\sqrt{5+\pi^2}} = \ldots$$

Torsion am $P = (2\pi, 0, -\pi) \rightarrow t = \pi$

$$\tau = \frac{(\dot{x}\ \ddot{x}\ \dddot{x})}{\|\dot{x}\|^2 \|\ddot{x}\|^2 - \langle \dot{x}, \ddot{x} \rangle^2} \qquad (*)$$

$$\dddot{x} = \begin{pmatrix} 0 \\ -2\sin t - \sin t - t\cos t \\ -2\cos t - \cos t + t\sin t \end{pmatrix} \qquad \dddot{x}(\pi) = \begin{pmatrix} 0 \\ \pi \\ +3 \end{pmatrix}$$

$$(\dot{x}, \ddot{x}, \dddot{x}) = \left| \begin{pmatrix} 2 & 0 & 0 \\ -\pi & -2 & \pi \\ -1 & \pi & 3 \end{pmatrix} \right| = 2(-6 - \pi^2) = -2(6+\pi^2)$$

In $(*)$: $\tau(\pi) = \left(-2(6+\pi^2)\right) \Big/ \sqrt{(5+\pi^2)(4+\pi^2) - \pi^2}$ //