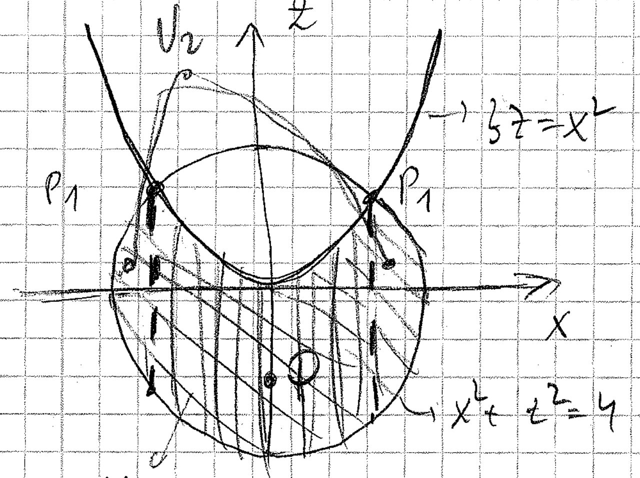
2) $x^2+y^2+z^2=4$

$x^2+y^2=3z$

$P(0,0,-1)$

Schnitt mit der xoz Ebene



P_1: $\left.\begin{array}{l} x^2+z^2=4 \\ x^2=3z \end{array}\right\} \Rightarrow 3z+z^2=4$

$z^2+3z-4=0$

$z=\dfrac{-3\pm\sqrt{9+16}}{2} \begin{cases} 1 \\ -4 \end{cases}$

$z=-4$ wird ausgeschl. weil außerhalb des Kreises $x^2+z^2=4$.

$z=1 \Rightarrow x^2=3z \Rightarrow 3 \Rightarrow x=\pm\sqrt{3}$

D.h. $P_1=(\sqrt{3},0,1)$ $P_2=(-\sqrt{3},0,1)$

Zylinderkoordinaten 6 $\quad x=r\cos\varphi$, $y=r\sin\varphi$, $z=z$, $dV=r\,dr\,d\varphi\,dz$

Gleichungen der Kugel u. des Paraboloids: $r^2+z^2=4$, $3z=r^2$

Grenzen: $0\le\varphi\le 2\pi$ $\quad 0\le r\le 2$

Für $0\le r\le\sqrt{3}$: $-\sqrt{4-r^2}\le z\le\dfrac{r^2}{3}$

Für $3\le r\le 2$ $\quad -\sqrt{4-r^2}\le z\le\sqrt{4-r^2}$

$$V=V_1+V_2=\int_{\varphi=0}^{2\pi}\int_{r=0}^{\sqrt{3}}\int_{z=-\sqrt{4-r^2}}^{r^2/3} r\,dz\,dr\,d\varphi+\int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\varphi$$

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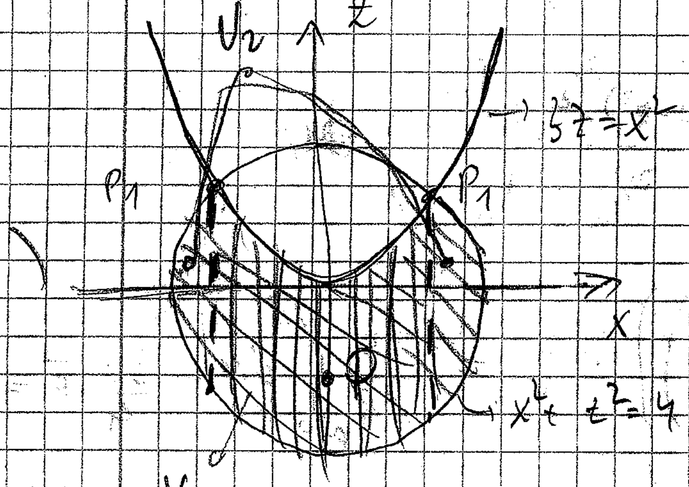

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$$V=V_1+V_2=\int_{\varphi=0}^{2\pi}\int_{r=0}^{\sqrt{3}}\int_{z=-\sqrt{4-r^2}}^{r^2/3} r\,dz\,dr\,d\varphi+\int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\varphi$$

$$V_1 = \int_{\varphi=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \int_{-\sqrt{4-r^2}}^{\frac{r^2}{3}} r\, dz\, dr\, d\varphi = \int_{\varphi=0}^{2\pi} \int_{r=0}^{\sqrt{3}} r \cdot z \Big|_{-\sqrt{4-r^2}}^{r^2/3} dr\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{r=0}^{\sqrt{3}} r\left(\frac{r^2}{3} + \sqrt{4-r^2}\right) dr\, d\varphi = \int_{\varphi=0}^{2\pi} \frac{1}{2} \int_{r=0}^{\sqrt{3}} \left(\frac{r^2}{3} + \sqrt{4-r^2}\right) d(r^2)\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \frac{1}{2}\left(\frac{(r^2)^2}{6}\Big|_0^{\sqrt{3}} - \frac{(4-r^2)^{3/2}}{3/2}\Big|_0^{\sqrt{3}} \right) d\varphi =$$

$$= \int_{\varphi=0}^{2\pi} \left(\frac{9}{18} - \frac{1}{2}\left(\frac{2}{3} - \frac{2 \cdot 8}{3}\right)\right) d\varphi = \frac{33}{18} \int_{\varphi=0}^{2\pi} d\varphi = \frac{33}{9} \cdot \pi = 3\frac{2}{3}\pi$$

$$V_2 = \int_{\varphi=0}^{2\pi} \int_{r=\sqrt{3}}^{2} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\varphi = \int_{\varphi=0}^{2\pi} \int_{r=\sqrt{3}}^{2} r \cdot z \Big|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dr\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \int_{r=\sqrt{3}}^{2} 2r\sqrt{4-r^2}\, dr\, d\varphi = \int_{\varphi=0}^{2\pi} - \int_{r=\sqrt{3}}^{2} \sqrt{4-r^2}\, d(-r^2)\, d\varphi =$$

$$= \int_{\varphi=0}^{2\pi} - \frac{(4-r^2)^{3/2}}{3/2}\Big|_{\sqrt{3}}^{2} d\varphi = \int_{\varphi=0}^{2\pi} \frac{2}{3} d\varphi = \frac{2}{3} \cdot 2\pi = \frac{4}{3}\pi = 1\frac{1}{3}\pi$$

$$V = V_1 + V_2 = 3\frac{2}{3}\pi + 1\frac{1}{3}\pi = 5\pi$$

③ $P_1 = \left(\pi, \frac{\pi}{2}, 0\right)$ $P_2 = \left(-\pi, \frac{\pi}{2}, 0\right)$ $P = (2\pi, 0, -\pi)$

$$x(t) = \begin{pmatrix} 2t \\ t \sin t \\ t \cos t \end{pmatrix}$$

$$\dot{x}(t) = \begin{pmatrix} 2 \\ \sin t + t \cos t \\ \cos t - t \sin t \end{pmatrix}$$

$$\|\dot{x}(t)\|^2 = 4 + \sin^2 t + t^2 \cos^2 t + \cancel{2t \sin t \cos t} + \cos^2 t + t^2 \sin^2 t$$
$$\cancel{- 2t \cos t \sin t} = 5 + t^2$$

$$L = \int_{P_1}^{P_2} dl = \int_{t=+\frac{\pi}{2}}^{-\frac{\pi}{2}} \|\dot{x}(t)\| \, dt = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{5+t^2} \, dt =$$

$$x\left(\frac{\pi}{2}\right) = \begin{pmatrix} 2 \cdot \frac{\pi}{2} \\ \frac{\pi}{2} \sin \frac{\pi}{2} \\ \frac{\pi}{2} \cos \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} \pi \\ \frac{\pi}{2} \\ 0 \end{pmatrix} = P_1$$

$$x\left(-\frac{\pi}{2}\right) = \begin{pmatrix} 2\left(-\frac{\pi}{2}\right) \\ -\frac{\pi}{2} \sin\left(-\frac{\pi}{2}\right) \\ -\frac{\pi}{2} \cos\left(-\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} -\pi \\ \frac{\pi}{2} \\ 0 \end{pmatrix} = P_2$$

$$= -2 \int_0^{\frac{\pi}{2}} \sqrt{5+t^2} \, dt$$

Variablensubstitution: $t = \sinh u$ $dt = \cosh u \, du$

$1 + t^2 = 1 + \sinh^2 u = \cosh^2 u$

$\operatorname{arsinh}\left(\frac{\pi}{2}\right) = a$ $\sqrt{1+t^2} = \sqrt{\cosh^2 u} = \cosh u$

$$L = -\int \cosh^2 u \, du = -\left[\sinh u \cosh u \Big|_0^a - \int_0^a \sinh^2 u \, du\right] =$$

$\operatorname{arsinh}(0) = 0$

part. Int.
$x' = \cosh u$
$y = \cosh u$

$$= -\left[\sinh u \cosh u \Big|_0^a - \int_0^a (\cosh^2 u - 1) \, du\right]$$

$$= -\left[\sinh a \cosh a + a - \int_0^a \cosh^2 u \, du\right]$$

Daraus folgt:

$$2\int_0^a \cosh^2 u \, du = \sinh a \cosh a + a$$

$$\Downarrow$$

$$\int_0^a \cosh^2 u \, du = \frac{\sinh a \cosh a + a}{2}$$

Somit gilt $L = -\dfrac{\sinh a \cosh a + a}{2}$

$L = -2.0792$ (negativ weil die Kurve negativ orientiert von P_2 mit $t = \frac{\pi}{2}$ nach P_1 mit $t = -\frac{\pi}{2} < \frac{\pi}{2}$)

Länge im physikalischen Sinne $= 2.0792$

Krümmung in P: $k = \dfrac{\sqrt{\|\dot{x}\|^2 \|\ddot{x}\|^2 - \langle \dot{x}, \ddot{x} \rangle^2}}{\|\dot{x}\|^3}$

$P = (2\pi, 0, -\pi)$

$$\ddot{x} = \begin{pmatrix} 0 \\ \cos t + \cos t - t \sin t \\ -\sin t - \sin t - t\cos t \end{pmatrix}$$

$$\dot{x}(2\pi) = \begin{pmatrix} 2 \\ 2\pi \\ 1 \end{pmatrix} \qquad \ddot{x}(2\pi) = \begin{pmatrix} 0 \\ 2 \\ -2\pi \end{pmatrix}$$

$\|\dot{x}(2\pi)\|^2 = 5 + 4\pi^2 \quad \|\ddot{x}(2\pi)\|^2 = 4 + 4\pi^2 = 4(1+\pi^2)$

$\langle \dot{x}, \ddot{x} \rangle = (4\pi - 2\pi) = 2\pi$

$$k(2\pi) = \frac{\sqrt{(5+4\pi^2)\,4(1+\pi^2) - 4\pi^2}}{(5+4\pi^2)^{3/2}} = \frac{\sqrt{4\pi^4 + 36\pi^2 + 20 - 4\pi^2}}{(5+4\pi^2)^{3/2}}$$

$$k(2\pi) = \frac{2\sqrt{\pi^4 + 8\pi^2 + 5}}{(5+4\pi^2)^{3/2}} \approx 0.0408$$

$$\tau(2\pi) = \left. \frac{(\dot{x}, \ddot{x}, \dddot{x})}{\|\dot{x}\|^2 \|\ddot{x}\|^2 - \langle \dot{x}, \ddot{x} \rangle^2} \right|_{t=2\pi}$$

$$\dddot{x} = \begin{pmatrix} 0 \\ -2\sin t - \sin t - t\cos t \\ -2\cos t - \cos t + t\sin t \end{pmatrix} \quad \dddot{x}(2\pi) = \begin{pmatrix} 0 \\ -2\pi \\ -3 \end{pmatrix}$$

$$(\dot{x}, \ddot{x}, \dddot{x})_{t=2\pi} = \left| \begin{pmatrix} 2 & 0 & 0 \\ 2\pi & 2 & -2\pi \\ 1 & -2\pi & -3 \end{pmatrix} \right| = 2(-3 - 4\pi^2)$$
$$= -2(4\pi^2 + 3)$$

$$\tau(\pi) = \frac{-2(4\pi^2+3)}{(5+4\pi^2)4(1+\pi^2) - 4\pi^2} = \frac{\cancel{2}(4\pi^2+3)}{\cancel{4}_2(\pi^4 + 8\pi^2 + 5)}$$

$$\tau(2\pi) = -0.2342$$