Game Theory WS 2013/2014

12. Exercise Sheet

38. **Sequencing game**
Don, a painter, is hired to paint the houses of Henry, Ethan and Tom. The following table depicts the sequential ordering of each client in Don's schedule book, the amount of (working) days required to paint his house, and the loss he suffers from every day that passes while work is uncompleted.

Sequetial ordering	Name	Time required	Daily loss in dollars
1	Henry	5	200
2	Ethan	3	550
3	Tom	4	400

Write down the coalitional function of the corresponding sequencing game in which the worth of a coalition is the sum of money that the members of the coaliotion can save by changing their ordering in a *feasible* way in Don's schedule book (cf. to the lecture for the definition of a *feasible* way of changing the ordering).

39. (a) Check that the following weighted majority games share the same coalitional function and therefore are different representations of the same game:

$$[2; 1, 1, 1]\,,\;\; [9; 8, 2, 7]\,,\;\; [9; 8, 1, 8].$$

(b) The representation $[2; 1, 1, 1]$ from (a) has the property that the sum of the weights equals the quota 2 in every minimal winning coalition. (A minimal winning coalition is a winning colaition such that every one of its proper subsets is not winning.) Such weights are called *homogeneous weights*, and the corresponding representation is called a *homogeneous representation*. In general the weights $w_1, w_2, \ldots, w_n$ are called *homogeneous weights*, iff there exists a real number q such that in the weighted majority game $[q; w_1, \ldots, w_n]$ the equality $q = \sum_{i \in S} w_i$ holds for every minimal winning coalition. This representation is then called a *homogeneous representation*.

For each of the two following weighted majority games determine whether it has a homogeneous representation. If yes, write it down. If no, explain why:

$$[10; 9, 1, 2, 3, 4]\,,\;\; [8; 5, 4, 2].$$

40. Prove the following theorem. Let $(N; v)$ be a coalitional game. Then

(a) $(N; v)$ is strategically equivalent to a 0-1 normalized game if and only if $v(N) > \sum_{i \in N} v(i)$.

(b) $(N; v)$ is strategically equivalent to a 0-0 normalized game if and only if $v(N) = \sum_{i \in N} v(i)$.

(c) $(N; v)$ is strategically equivalent to a 0-(-1) normalized game if and only if $v(N) < \sum_{i \in N} v(i)$.

41. Let $(N; v)$ be a coalitional game with a set of players $N = \{1, 2, 3\}$ and coalitional function

$$v(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S = \{1\} \text{ or } S = \{2\} \\ 2 & \text{if } S = \{3\} \\ 4 & \text{if } |S| = 2 \\ 5 & \text{if } |S| = 3 \end{cases}$$

Is $(N; v)$ a superadditive game? What is the set of imputations of this game?

42. Give the example of a monotonic game that is not superadditive, and an example of a superadditive game, that is not monotonic.

43. (a) Is the three-player game $(N;v)$ with $N = \{1,2,3\}$ and $v(1) = 3$, $v(2) = 13$, $v(3) = 4$, $v(1,2) = 12$, $v(1,3) = 15$, $v(2,3) = 1$ and $v(1,2,3) = 10$ monotonic?
    (b) Find a monotonic game that is strategically equivalent to $(N;v)$.
    (c) Prove that every game is strategically equivalent to a monotonic game. Follow from there that monotonicity is not invariant under strategic equivalence.

44. Prove that for every coalitional game $(N;v)$ there exists a coalitional structure $\mathcal{B}$, for which the set of imputations $\mathcal{X}(\mathcal{B}, v)$ is non-empty.

45. Let $(N;v)$ and (N,w) be two coalitional games with the same set of players. Let $x \in \mathcal{X}(\mathcal{B}, v)$ and $y \in \mathcal{X}(\mathcal{B}, w)$ for some coalitional structure $\mathcal{B}$. Does $x + y \in \mathcal{X}(\mathcal{B}, v + w)$ necessarily hold? Does $x - y \in \mathcal{X}(\mathcal{B}, v - w)$ necessarily hold? If you answer yes to either question, then provide a proof. If you answer no, present a counterexample.

46. (a) Write down the set of imputations in the three player game $(N;v)$ with $N = \{1,2,3\}$ and $v(1) = 3$, $v(2) = 5$, $v(3) = 7$, $v(1,2) = 6$, $v(1,3) = 12$, $v(2,3) = 15$ and $v(1,2,3) = 10$ for all coalitional structures.
    (b) Repeat (a) if $v(1,2,3) = 13$.
    (c) Repeat (a) if $v(1,2,3) = 34$.

47. Give an example of a monotonic game with an empty core and an example of a superadditive game with an empty core.