Risk theory and risk management in actuarial science
Winter term 2016/17

5th work sheet

28. **Archimedian Copulas**

(a) Show that for every $\theta \in \mathbb{R}$ the function $\phi_\theta^{Fr}(t) = -\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$ generates an Archmedian copula, the so-called Frank copula $C_\theta^{Fr}\colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1, u_2 \in [0,1]$:

$$C_\theta^{Fr}(u_1,u_2) = -\frac{1}{\theta}\ln(1+\frac{(\exp(-\theta u_1)-1)(\exp(-\theta u_2)-1)}{\exp(-\theta)-1}, \theta \in \mathbb{R}.$$

(b) Show that for every $\theta \geq 0$ and for every $\delta \geq 1$ the function $\phi_{\theta,\delta}^{GC}(t) = \theta^{-\delta}(t^{-\theta}-1)^\delta$ generates an Archmedian copula, the so-called *generalized Clayton copula* $C_{\theta,\delta}^{GC}\colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1, u_2 \in [0,1]$:

$$C_{\theta,\delta}^{GC}(u_1,u_2) = \{[(u_1^{-\theta}-1)^\delta + (u_2^{-\theta}-1)^\delta]^{1/\delta}+1\}^{-1/\theta},\ \theta \geq 0, \lambda \geq 1.$$

(c) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependency for the copulay C_θ^{Gu}, C_θ^{Cl}, C_θ^{Fr} and $C_{\theta,\delta}^{GC}$, respectively, and summarize the results in a table. (The coefficients which have been computed in the lecture do not need to be recomputed).

(d) Show that C_θ^{Gu} tends to the independence copula Π if θ tends to 1 and to the upper Fréchet bound M if θ tends to infinity. In this case we say that *the lower limit of the Gumbel copula is the independence copula* Π *and its upper limit is the Fréchet upper bound* M. Analogously show that the lower limit of the Clayton copula is the Fréchet lower bound W and its upper limit is the Fréchet upper bound M. Finally show that the Frank copula has the same lower and upper limits as the Clayton copula, respectively.

29. (a) Let $(X_1, X_2)^T$ be a t-distributed random vektor with ν degrees of freedom, expected value $(0,0)$ and linear correlation coefficient matrix ρ: $(X_1,X_2)^T \sim t_2(\vec{0},\nu,R)$ where R is 2×2 matrix with 1 on the diagonal and ρ outside the diagonal. Show that the following equality holds for $\rho > -1$:

$$\lambda_U(X_1,X_2) = \lambda_L(X_1,X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$$

Hint: Use the fact (no need to prove it!) that the X_2 conditioned upon $X_1 = x$ has a t-distribution as follows

$$X_2|X_1 = x \sim \left(\frac{\nu+1}{\nu+x^2}\right)^{1/2}\frac{X_2-\rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}.$$

Moreoever use the stochatic representation of the t-distribution of (X_1,x_2) as $\mu + \sqrt{W}AZ$, where Z is bivariate standard normally distributed and W is such that $\frac{\mu}{/}W \sim \}chi_\nu^2$ while being independent on Z, cf. lecture.

(b) Apply (a) to conclude that for a random vector with continuous marginal distributions $(X_1,X_2)^T$ and a t-copula $C_{\nu,R}^t$ with ν degrees of freedom and a correlation matrix R as in (a) the following equalities holds:

$$\lambda_U(X_1,X_2) = \lambda_L(X_1,X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right).$$

30. A bank has a loan portfolio of 100 loans. Let X_k be the default indicator for loan k such that $X_k = 1$ in case of default and 0 otherwise, for $k \in \{1,\ldots,100\}$.

(a) Supoose that X_k are independent and identically distributed with $P(X_k = 1) = 0.01$. Compute the expected value $E(N)$ of the number N of defaults and $P(N = k)$ for $k \in \{0, 1, \ldots, 100\}$.

(b) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z$, where $P(Z = 0.01) = 0.9$ and $P(Z = 0.11) = 0.1$. Compute the expected value $E(N)$ where N is defined as in (a).

(c) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z^9$, where Z is uniformly distributed on $(0, 1)$. Compute the expected value $E(N)$ where N is defined as in (a).