Risk theory and risk management in actuarial science
Winter term 2016/17

1st work sheet

1. Let $L \sim N(\mu, \sigma^2)$. Show that $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma\Phi^{-1}(\alpha)$ holds, where Φ is the distribution function of a random variable $X \sim N(0,1)$. Further show that $CVaR_\alpha(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds, where ϕ is the density function of X as above.

2. Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d.: $X_1 = \ln\frac{S_1}{S_0}$, $X_2 = \ln\frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).

   (a) Compute $VaR_{0.99}(L_1)$.

   (b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L_{100}^\Delta)$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. L_{100}^Δ is the linearization of the above mentioned 100-day PF-portfolio loss. Compare the two values and comment on the results.

   Hint: Use the equality $\Phi^{-1}(0.99) \approx 2.3$, where Φ is the distribution function of a random variable $X \sim N(0,1)$.

3. (a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

   (b) Let the distribution function F_L of the loss function L be given by $F_L(x) = 1 - (1+\gamma x)^{-1/\gamma}$ for $x \geq 0$ and some parameter $\gamma \in (0,1)$ (this is the generalized Pareto distribution). Compute $CVaR_\alpha(L)$.

4. Let the loss L be distributed according to the Students t-distribution with $\nu > 1$ degrees of freedom. The density function of L is given as

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

   Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\left(\frac{\nu + (t_\nu^{-1}(a))^2}{\nu - 1}\right)$, where t_ν is the distribution function of L.

5. Show that the following distributions are regularly varying:

   (a) The Pareto distribution G_α with parameter α given as $G_\alpha(x) = 1 - x^{-\alpha}$, for $x > 1$, where $\alpha > 0$. Show that $\bar{G}_\alpha(tx)/\bar{G}_\alpha(x) = x^{-\alpha}$ holds for $t > 0$, thus $\bar{G}_\alpha \in RV_{-\alpha}$.

   (b) The Fréchet distribution Φ_α with parameter α given as $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\Phi_\alpha(0) = 0$, where $\alpha > 0$. Show that $\lim_{x\to\infty} \bar{\Phi}_\alpha(x)/x^{-\alpha} = 1$, i.e. $\bar{\Phi}_\alpha \in RV_{-\alpha}$.

6. Show that for any distribution F, $F \in DA(G_2)$ if and only if $F \in DA(\Phi)$, where Φ is the standard normal distribution, $\Phi \sim N(0,1)$, DA stands for "Domain of attraction", and G_2 is the stable distribution with form parameter $\alpha = 2$ (and arbitrary parameters β and c).

   Hint: Apply the *Convergence to types theorem*.

7. Prove the following characterization of the maximum domain of attraction of an extreme value distribution H (also fomulated in the lecture):
   $F \in MDA(H)$ with normalizing and centralizing constants $a_n > 0$, b_n, $n \in \mathbb{N}$, respectively, iff $\lim_{n\to\infty} n\bar{F}_n(a_n x + b) = -\ln(H(x))$, for all $x \in \mathbb{R}$.

8. (Poisson distribution)
Let $X \sim P(\lambda)$, i.e. $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, for some parameter $\lambda > 0$. Show that there exists no extreme value distribution Z such that $X \in MDA(Z)$.

   Hint: Use Leadbetter et al.'s Lemma as follows (cf. lecture). For any discrete non-negative distribution F with right end $x_F = +\infty$ (i.e. a random variable with distribution F can take arbitrarily large values), the following two statements are equivalent for every $\tau \in (0, \infty)$: a) there exists a sequence $u_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} n\bar{F}(u_n) = \tau$, and b) $\lim_{n\to\infty} \frac{\bar{F}(n)}{\bar{F}(n-1)} = 1$.

   You don't need to prove the lemma.

9. (Maximum domain of attraction of the Fréchet distribution)
Show that the following distributions belong to the maximum domain of attraction $MDA(\Phi_\alpha)$ of the Fréchet distribution Φ_α, for some $\alpha > 0$, and determine the normalizing and centralizing constants $a_n > 0$, b_n, for $n \in \mathbb{N}$, respectively.

   (a) The Pareto distribution with parameter $\alpha > 0$: $G_\alpha(x) = 1 - x^{-\alpha}$, for $x > 1$.
   (b) The Cauchy distribution with density function $f(x) = (\pi(1 + x^2))^{-1}$, $x \in \mathbb{R}$.
   (c) The Students distribution with parameter $\alpha \in \mathbb{N}$ and density function $f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)(1+x^2/\alpha)^{(\alpha+1)/2}}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.
   (d) The Loggamma distribution with paramters $\alpha, \beta > 0$ and density function $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1}x^{-\alpha-1}$, for $x > 1$.

10. (Maximum domain of attraction of the Weibull ditribution)
Let $X \sim U(0, 1)$ be uniformly distributed on $[0, 1]$. Show that X belongs to the maximum domain of attraction of the Weibull distribution with parameter 1, i.e. $X \in MDA(\Psi_1)$, with normalizing constant $a_n = 1/n$, for $n \in \mathbb{N}$.

11. (Maximum domain of attraction of the Gumbel ditribution)
Check whether the following distributions belong to the maximum domain of attraction $MDA(\Lambda)$ of the Gumbel distribution.

   (a) The normal distribution $F(x) = (2\pi)^{-1/2}\exp\{-x^2/2\}$, $x \in \mathbb{R}$.
   (b) The exponential distribution with density function $f(x) = \lambda^{-1}\exp\{-\lambda x\}$, $x > 0$, for some parameter $\lambda > 0$.
   (c) The lognormal distribution with density function $f(x) = (2\pi x^2)^{-1/2}\exp\{-(\ln x)^2/2\}$, $x > 0$.
   (d) The gamma distribution with density function $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp\{-\beta x\}$, $x > 0$, for some parameters $\alpha, \beta > 0$.