Example 5 *European call option (ECO)*

Consider an ECO over an asset S with *execution date* T, price S_T at time T and *strike price* K.

Value of the ECO at time T: $\max\{S_T - K, 0\}$

Price of ECO at time $t < T$: $C = C(t, S, r, \sigma)$ (Black-Scholes model), where S is the price of the asset, r is the interest rate and σ is the volatility, all of them at time t.

Risk factors: $Z_n = (\ln S_n, r_n, \sigma_n)^T$;

Risk factor changes: $X_{n+1} = (\ln S_{n+1} - \ln S_n, r_{n+1} - r_n, \sigma_{n+1} - \sigma_n)^T$

Portfolio value: $V_n = C(t_n, S_n, r_n, \sigma_n) = C(t_n, exp(Z_{n,1}), Z_{n,2}, Z_{n,3})$

The linearized loss: $L^{\Delta}_{n+1} = -(C_t \Delta t + C_S S_n X_{n+1,1} + C_r X_{n+1,2} + C_\sigma X_{n+1,3})$

The greeks: C_t - theta, C_S - delta, C_r - rho, C_σ - Vega

Purpose of the risk management:

- Determination of the minimum regulatory capital:
  i.e. the capital, needed to cover possible losses.

- As a management tool:
  to determine the limits of the amount of risk a unit within the company may take

Some basic risk measures (not based on the loss distribution)

- Notational amount: weighted sum of notational values of individual securities weighted by a prespecified factor for each asset class

  e.g. in Basel I (1998):

  $$\textit{Cooke Ratio} = \frac{\text{regulatory capital}}{\text{risk-weighted sum}} \geq 8\%$$

  $$\text{Gewicht} := \begin{cases} 0\% & \text{for claims on governments and supranationals (OECD)} \\ 20\% & \text{claims on banks} \\ 50\% & \text{claims on individual investors with mortgage securities} \\ 100\% & \text{claims on the private sector} \end{cases}$$

  Disadvantages: no difference between long a short positions, does not consider diversification effects

- Coefficients of sensitivity with respect to risk factors

  Portfolio value at time t_n: $V_n = f(t_n, Z_n)$,
  Z_n ist a Vektor of d risk factors

  Sensitivity coefficients: $f_{z_i} = \frac{\delta f}{\delta z_i}(t_n, Z_n)$, $1 \le i \le d$

  Example: "The Greeks" of a portfolio are the sensitivity coefficients

  Disadvantages: Assessment of risk arising due to simultaneous change of different risk factors is difficult;
  aggregation of risks arising in differnt markets is difficult ;

- Scenario based risk measures: Let n be the number of possible risk factor changes (= szenarios).

  Let $\chi = \{X_1, X_2, \ldots, X_N\}$ be the set of scenarios and $l_{[n]}(\cdot)$ the portfolio loss operator.

  Assign a weight w_i to every scenario i, $1 \le i \le N$

  Portfolio risk:

  $\Psi[\chi, w] = \max\{w_1 l_{[n]}(X_1), w_2 l_{[n]}(X_2), \ldots, w_N l_{[n]}(X_N)\}$

Example 6 *SPAN ruled applied at CME* (see Artzner et al., 1999)

Portfolio consists of many units of a certain future contract and many put and call options on the same contract with the same maturity.

Computing SPAN Marge:

Scenarios i, $1 \leq i \leq 14$:

Scenarios 1 to 8		Scenarios 9 to 14	
Volatility	Price of the future	Volatility	Price of the future
$\nearrow$ $\searrow$	$\nearrow \frac{1}{3} * Range$ $\nearrow \frac{2}{3} * Range$ $\nearrow \frac{3}{3} * Range$ $\longrightarrow$	$\nearrow$ $\searrow$	$\searrow \frac{1}{3} * Range$ $\searrow \frac{2}{3} * Range$ $\searrow \frac{3}{3} * Range$

Scenarios i, $i = 15, 16$ represent an extreme increase or decrease of the future price, respectively.

$$w_i = \begin{cases} 1 & 1 \leq i \leq 14 \\ 0,35 & 15 \leq i \leq 16 \end{cases}$$

An appropriate model (zB. Black-Scholes) is used to generate the option prices in the different scenarios.

- **Risk measures based on the loss distribution**

  Let $F_L := F_{L_{n+1}}$ be the loss distribution of L_{n+1}.

  The parameter of F_L will be estimated in terms of historical data, either ddirectly oder by involving risk factors.

  1. **The standard deviation** $std(L) := \sqrt{\sigma^2(F_L)}$
     It is used frequently in portfolio theory.

     Disadvantages:

     – STD exists only for distributions with $E(F_L^2) < \infty$, not applicable to leptocurtic ("fat tailed") loss distributions;

     – gains and losses equally influence the STD.

     Example 7 $L_1 \sim N(0, 2)$, $L_2 \sim t_4$ *(Student's distribution with 4 degrees of freedom)*
     $\sigma^2(L_1) = 2$ *and* $\sigma^2(L_2) = \frac{m}{m-2} = 2$ *hold, where* m *is the number of degrees of freedom, thus* $m = 2$*.*
     However the probability of losses is much larger for L_2 *than for* L_1*.*
     Plot the logarithm of the quotient $\ln[P(L_2 > x)/P(L_1 > x)]$*!*

2. **Value at Risk** $(VaR_\alpha(L))$

Definition 5 *Let L be the loss distribution and $\alpha \in (0,1)$ a given confindence level.*

$VaR_\alpha(L)$ is the smallest number l, such that $P(L > l) \leq 1 - \alpha$ holds.

$VaR_\alpha(L) = \inf\{l \in \mathbb{R} \colon P(L > l) \leq 1 - \alpha\} =$
$\inf\{l \in \mathbb{R} \colon 1 - F_L(l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R} \colon F_L(l) \geq \alpha\}$

BIS (Bank of International Settlements) suggests $VaR_{0.99}(L)$ over a horizon of 10 days as a measure for the market risk of a portfolio.

Definition 6 *Let $F \colon A \to B$ be a monotone increasing function (d.h. $x \leq y \Longrightarrow F(x) \leq F(y)$). The function*

$$F^{\leftarrow} \colon B \to A \cup \{-\infty, +\infty\}, y \mapsto \inf\{x \in \mathbb{R} \colon F(x) \geq y\}$$

is called generalized inverse function of F.

Notice that $\inf \emptyset = \infty$.

If F is strictly monotone increasing, then $F^{-1} = F^{\leftarrow}$ holds.

Exercise 1 *Compute $F^{\leftarrow}$ for $F \colon [0, +\infty) \to [0,1]$ with*

$$F(x) = \begin{cases} 1/2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

Definition 7 *Let $F\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. $q_\alpha(F) := \inf\{x \in \mathbb{R}\colon F(x) \geq \alpha\}$ is called α-quantile of F.*

For the loss function L and its distribution function F the following holds:

$$VaR_\alpha(L) = q_\alpha(F) = F^{\leftarrow}(\alpha).$$

Example 8 *Let $L \sim N(\mu, \sigma^2)$.*
Then $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma\Phi^{-1}(\alpha)$ holds, where Φ is the distribution function of a random variable $X \sim N(0,1)$.

Exercise 2 *Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d.: $X_1 = \ln \frac{S_1}{S_0}$, $X_2 = \ln \frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).*

(a) Compute $VaR_{0.99}(L_1)$.

(b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L^{\triangle}_{100})$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. $L^{\triangle}_{100}$ is the linearization of the above mentioned 100-day PF-portfolio loss.

Hint: For $Z \sim N(0,1)$ use the equality $F_Z^{-1}(0.99) \approx 2.3$.

3. **Conditional Value at Risk** ($CVaR_\alpha(L)$) (or *Expected Shortfall* (ES))

A disadvantage of VaR: It tells nothing about the amount of loss in the case that a large loss $L \geq VaR_\alpha(L)$ happens.

Definition 8 *Let α be a given confidence level and L a continuous loss function with distribution function F_L. $CVaR_\alpha(L) := ES_\alpha(L) = E(L|L \geq VaR_\alpha(L))$.*

If F_L is continuous:

$CVaR_\alpha(L) = E(L|L \geq VaR_\alpha(L)) = \frac{E(LI_{[q_\alpha(L),\infty)}(L))}{P(L \geq q_\alpha(L))} = \frac{1}{1-\alpha}E(LI_{[q_\alpha(L),\infty)}) = \frac{1}{1-\alpha}\int_{q_\alpha(L)}^{+\infty} l dF_L(l)$

I_A is the indicator function of the set A: $I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

If F_L is discrete the *generalized CVaR* is defined as follows:

$$GCVaR_\alpha(L) := \frac{1}{1-\alpha}\left[E(LI_{[q_\alpha(L),\infty)}) + q_\alpha\Big(1 - \alpha - P(L > q_\alpha(L))\Big)\right]$$

Lemma 1 *Let α be a given confidence level and L a continuous loss function with distribution F_L.*
Then $CVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_p(L)dp$ holds.

Example 9 *(a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.*

(b) Let the distribution function F_L of the loss function L be given as follows : $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and $\gamma \in (0,1)$. Compute $CVaR_\alpha(L)$.

Example 10 *Let $L \sim N(0,1)$. Let ϕ und Φ be the density and the distribution function of L, respectively. Show that $CVaR_\alpha(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.*

Let $L' \sim N(\mu, \sigma^2)$. Show that $CVaR_\alpha(L') = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

Exercise 3 *Let the loss L be distributed according to the Student's t-distribution with $\nu > 1$ degrees of freedom. The density of L is*

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\Big(1 + \frac{x^2}{\nu}\Big)^{-(\nu+1)/2}$$

Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\left(\frac{\nu+(t_\nu^{-1}(a))^2}{\nu-1}\right)$, where t_ν is the distribution function of L.

Methods for the computation of VaR und CVaR

Consider the portfolio value $V_m = f(t_m, Z_m)$, where Z_m is the vector of risk factors.

Let the loss function over the interval $[t_m, t_{m+1}]$ be given as $L_{m+1} = l_{[m]}(X_{m+1})$, where X_{m+1} is the vector of the risk factor changes, i.e.

$$X_{m+1} = Z_{m+1} - Z_m.$$

Consider observations (historical data) of risk factor values $Z_{m-n+1}, \dots, Z_m$.

How to use these data to compute/estimate $VaR(L_{m+1})$, $CVaR(L_{m+1})$?

The empirical VaR and the empirical CVaR

Let $x_1, x_2, \ldots, x_n$ be a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n$ with distribution function F

The empirical distribution functionis given as

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I_{[x_k, +\infty)}(x)$$

The empirical quantile is given as

$$q_\alpha(F_n) = \inf\{x \in \mathbb{R} \colon F_n(x) \geq \alpha\} = F_n^{\leftarrow}(\alpha)$$

Assumption: $x_1 > x_2 > \ldots > x_n$. Then $q_\alpha(F_n) = x_{[n(1-\alpha)]+1}$ holds, where $[y] := \sup\{n \in \mathbb{N} \colon n \leq y\}$ for every $y \in \mathbb{R}$.

Let $\widehat{q}_\alpha(F) := q_\alpha(F_n)$ be the empirical estimator of the quantile $q_\alpha(F)$.

Lemma 2 *Let F be a strictly increasing funkcion. Then $\lim_{n\to\infty} \widehat{q}_\alpha(F) = q_\alpha(F)$ holds $\forall \alpha \in (0,1)$, i.e. the estimator $\widehat{q}_\alpha(F)$ is consistent.*

The empirical estimator of CVaR is

$$\widehat{CVaR}_\alpha(F) = \frac{\sum_{k=1}^{[n(1-\alpha)]+1} x_k}{[(n(1-\alpha)] + 1}$$

A non-parametric bootstrapping approach to compute the confidence interval of the estimator

Let the random variables $X_1, X_2, \ldots, X_n$ be i.i.d. with distribution function F and let $x_1, x_2, \ldots x_n$ be a sample of F.

Goal: computation of an estimator of a certain parameter θ depending on F, e.g. $\theta = q_\alpha(F)$, and the corresponding confidence interval.

Let $\widehat{\theta}(x_1, \ldots, x_n)$ be an estimator of θ, e.g. $\widehat{\theta}(x_1, \ldots, x_n) = x_{[(n(1-\alpha)]+1,n}$ $\theta = q_\alpha(F)$, where $x_{1,n} > x_{2,n} > \ldots > x_{n,n}$ is the ordered sample.

The required confidence interval is an (a, b) with $a = a(x_1, \ldots, x_n)$ u. $b = b(x_1, \ldots, x_n)$, such that $P(a < \theta < b) = p$, for a given confidence level p.

<u>Case I</u>: F is known.

Generate N samples $\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}$, $1 \le i \le N$, by simulation from F (N should be large)

Let $\tilde{\theta}_i = \widehat{\theta}\big(\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}\big)$, $1 \le i \le N$.

A non-parametric bootstrapping approach to compute the confidence interval of the estimator

Case I (cont.)

The empirical distribution function of $\widehat{\theta}(x_1, x_2, \ldots, x_n)$ is given as

$$F_N^{\widehat{\theta}} := \frac{1}{N} \sum_{i=1}^{N} I_{[\widetilde{\theta}_i, \infty)}$$

and it tends to $F^{\widehat{\theta}}$ for $N \to \infty$.

The required conficence interval is given as $\left(q_{\frac{1-p}{2}}(F_N^{\widehat{\theta}}), q_{\frac{1+p}{2}}(F_N^{\widehat{\theta}})\right)$

(assuming that the sample sizes N und n are large enough).