Operations Research
Winter term 2015/2016
2nd work sheet (dynamic optimization)

9. A company produces a delicate product by using a sofisticated technology which is not fully developed yet. Every piece of the product works well with probability 1/2 and is irreparably defective with probability 1/2. The company produces many pieces of the product in each production run and hopes that at least one working piece will be produced. The remaining pieces are worthless and will be discarded no matter whether they are defective or not. The production costs amount to 100 Euro per piece and the fixed costs amount to 300 Euro per production run. The time available until delivery allows to complete at most three production runs. The company has to pay a fine of 1600 Euro if it cannot deliver a working piece of the product on due time. Determine a production policy, i.e. the number of the production runs and the number of pieces produced in each production run, such that the expected overall cost of the company is minimized.

10. A popular game in Las Vegas is the one called "all or nothing": in every run of the game the player places a number of chips and either wins all of them or loses all of them. Assume that the probability to win a run of the the game is 2/3 and different runs of the game are independent. Consider a player who possesses three chips, plays at most three runs, and considers the possession of five chips at the end of the game as a victory. Determine a playing strategy, which maximizes the probability of a victory (according to the players own definition of a victory). A playing strategy specifies the number of the placed chips in every run of the game depending on the outcome of the previous runs.

11. Consider a single-product warehouse with a storage capacity of M, $M \in \mathbb{N}$ (i.e. the warehouse can store at most M product units at a time). The warehouse gets inspected at discrete points n in time, $n = 0, 1, \ldots, N-1$, and at each inspection it will be decided whether to order a certain amount $b_n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ of the product and increase the stock or not. Assume that the ordered amount of the product reaches the warehouse immediately, i.e. at the very same moment in which the order has been placed. The demand on the product in the time interval $[n, n+1)$ is the realisation of a random variable Y_n, $n = 0, 1, \ldots, N-1$. The random variables Y_n, $n = 0, 1, \ldots, N-1$, are assumed to be identically and independently distributed with a density given as $P(Y_n = x) = q(x)$, $\forall x \in \mathbb{N}_0$ and $n = 0, 1, \ldots, N-1$. Moreover it is assumed that these random variables have a finite expected value. Assume further that the demand is fulfilled immediately after the arrival of the ordered products in the warehouse. If there is not enough stock to fulfill the demand in that moment, then the missed demand will be considered again, right after the arrival of the next order. The odering costs amount to cb_n for every placed order b_n, $n = 0, 1, \ldots, N-1$, where c is a given constant. There are also storage costs and shortfall costs given as

$$l(z_n) = \begin{cases} l_1 \cdot z & z \geq 0 \text{ (stock)} \\ -l_2 \cdot z & z < 0 \text{ (shortfall)} \end{cases},$$

where $l_1, l_2 \geq 0$ are prespecified constants, and z_n is the stock or shortfall immediately after the arrival of the order and the delivery of the demand at the beginning of the time interval $[n, n+1)$. The goal is to determine the minimum of the overall expected costs and an optimal ordering policy. Formulate this problem as a control problem and give the optimality equation. The decision variable should represent the amount of stock right after the arrival of the order and prior to the delivery.

12. Consider the situation in Exercise 11. Let $-V_n(s)$, $n = 0, 1, \ldots, N-1$, be the minimum expected overall costs starting at period n (until the end of the planing horizon), if the stock level is s at the beginning of the time interval $[n, n+1)$, immediately before the arrival of the order and the delivery. Define $G_n(s) := -V_n(s) + cs$, $n = 0, 1, \ldots, N-1$. Show that the equalities $G_n(s) = \min\{J_n(a)\colon s \leq a \leq M\}$, $n = 0, 1, \ldots, N-1$, hold, where

$$J_n(a) := c \cdot E + \sum_{x=0}^{\infty} q(x) l(a-x) + \sum_{x=0}^{\infty} q(x) G_{n+1}(a-x) \text{ für } n = 0, 1, \ldots, N-1,$$

and E is the expected value of the random variables Y_n as introduced in Exercise 11.

13. Let $v\colon S \to \mathbb{R}$ be a convex function. Define the functions

$$w_1(s) := \sum_{x \in \mathbb{N}_0} q(x)v(s-x) \text{ und } w_2(s) := \min\{v(a)\colon a \geq s\}\,.$$

in terms of a probability density function $q(x)$, $x \in \mathbb{N}_0$, as in Exercise 11. Show that w_1 and w_2 are convex functions.

14. Use the result of Exercise 13 to show that the functions $J_n(a)$ and $G_n(s)$, $n = 0, 1, \ldots, N-1$, defined as in Exercise 12, are convex. Deduce then that in a situation like in Exercise 11 there exists some thresholds $S_n^* \in \mathbb{N}_0$, $n = 0, 1, \ldots, N-1$, and an optimal strategy which places orders so as to achieve the level S_n^* of stock right after the arrival of the order in any period n, $n = 0, 1, \ldots, N-1$.