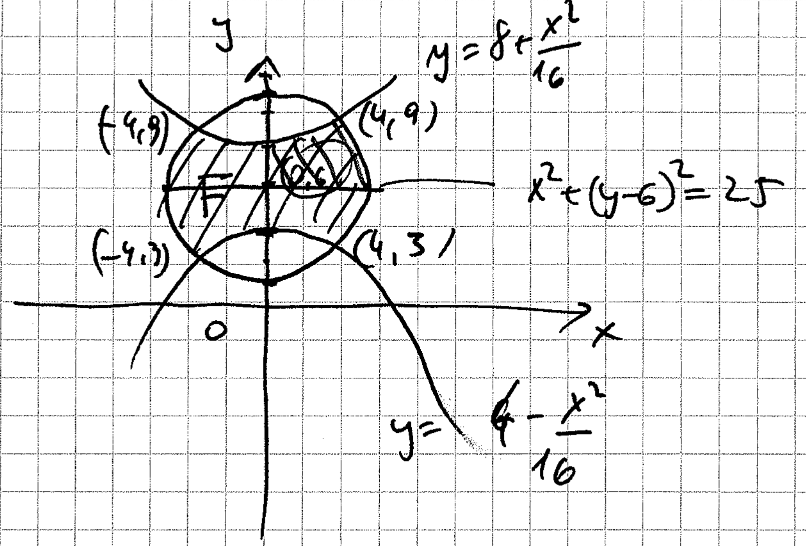
Nachklausur zur 8. Klausur

Bsp. 1



$$x^2+(y-6)^2=25$$

$$y=4-\frac{x^2}{16}$$

$$y=8+\frac{x^2}{16}$$

Guldin'sche Regel

Volum. eines Rot. körper = Fläche × Weg des Flächen Schw.pkt.

Fläche F wie im Bild.

Schnittpkt.:

$$\begin{cases} x^2+(y-6)^2=25 \\ y=4-\frac{x^2}{16} \end{cases} \Rightarrow \quad y-6=-2-\frac{x^2}{16}$$

$$(y-6)^2=\left(2+\frac{x^2}{16}\right)^2=4+\frac{x^4}{256}+\frac{4x^2}{16}$$

$$x^2+(y-6)^2=x^2+4+\frac{x^4}{256}+\frac{x^2}{4}=25$$

$$\Updownarrow$$

$$256x^2+256\cdot 4+x^4+64x^2-25\cdot 256=0$$

$$x^4+320x^2-256\cdot 21=0$$

$$x^2_{1,2}=\frac{-320\pm\sqrt{320^2+46\cdot 256}}{2} \begin{cases}\to 16 \checkmark \\ \to -336<0\end{cases}$$

$$x^2=16 \Rightarrow x=\pm 4$$

$$y=4-\frac{4^2}{16}=3$$

Schnittpkt. $(\pm 4, 3)$

Analog. die anderen Schnittpunkte:

$$\left.\begin{array}{l} x^2+(y-6)^2=25 \\ y=8+\frac{x^2}{16} \end{array}\right\} \Rightarrow \quad y-6=2+\frac{x^2}{16}$$

Dieselbe quadr. Gleich. wie beim ersten Schnitt!!

$$x^2=16 \Rightarrow x=\pm 4$$

$$y=8+\frac{16}{16}=9$$

Schnittpkt $(\pm 4, 9)$

$$F=4\left(\underbrace{\int_0^4\int_6^{8+\frac{x^2}{16}} dy\,dx}_{I_1}+\underbrace{\int_4^5\int_6^{\sqrt{25-x^2}+6} dy\,dx}_{I_2}\right)$$

aus Symmetrie-Gründen

$$I_1=\int_0^4\int_6^{8+\frac{x^2}{16}} dy\,dx=\int_0^4\left(8+\frac{x^2}{16}-6\right)dx=2x+\frac{x^3}{3\cdot 16}\Big|_0^4$$

$$I_1=8+\frac{64}{3\cdot 16}=8+\frac{4}{3}=\cancel{9}\frac{1}{3}=\boxed{\frac{28}{3}=I_1}$$

$$I_2=\int_4^5\int_6^{\sqrt{25-x^2}+6} dy\,dx=\int_4^5\sqrt{25-x^2}\,dx$$

$$I_2=5\int_4^5\sqrt{1-\left(\frac{x}{5}\right)^2}\,dx=25\int_4^5\sqrt{1-\left(\frac{x}{5}\right)^2}\,d\left(\frac{x}{5}\right)$$

$\frac{x}{5}=\sin t$

$t=\arcsin\left(\frac{x}{5}\right)$

$\cancel{=25\int}\ d\left(\frac{x}{5}\right)=\cos t\,dt$

$$\int\sqrt{1-\left(\frac{x}{5}\right)^2}\,d\left(\frac{x}{5}\right)=\int\sqrt{1-(\sin t)^2}\cos t\,dt=\int\cos^2 t\,dt$$

$$\int \cos^2 t\,dt = \cos t \sin t + \int \sin^2 t\,dt = \cos t \sin t + \int (1-\cos^2 t)\,dt$$

$$= \cos t \sin t + t - \int \cos^2 t\,dt$$

$\cos t = u \quad \cos t = v'$

$u' = -\sin t \quad v = \sin t$

$$\int \cos^2 t\,dt = \frac{\cos t \sin t + t}{2}$$

$$\int \sqrt{1-\left(\frac{x}{5}\right)^2}\,d\left(\frac{x}{5}\right) = \frac{\cos t \sin t + t}{2} = \frac{\frac{x}{5}\sqrt{1-\left(\frac{x}{5}\right)^2} + \arcsin\left(\frac{x}{5}\right)}{2}$$

$$I_2 = 25 \int_4^5 \sqrt{1-\left(\frac{x}{5}\right)^2}\,d\left(\frac{x}{5}\right) = 25 \left. \frac{\frac{x}{5}\sqrt{1-\left(\frac{x}{5}\right)^2} + \arcsin\left(\frac{x}{5}\right)}{2} \right|_4^5$$

$$I_2 = \frac{25}{2}\left(\frac{\pi}{2} - \frac{4}{5}\cdot\frac{3}{5} - \arcsin\left(\frac{4}{5}\right)\right) = 14.0712$$

$$I = 4\left(\frac{28}{3} + 14.0712\right) = 93.6181$$

Flächenschwerpkt: $(0, 6)$ aus Symmetriegründen

Weg. des Fl. Schw.pkt: $2\pi \cdot 6$

$$V = I \cdot 2\pi \cdot 6 = 12\pi I = 12\pi \cdot 93.6181 =$$

$$= \boxed{1123.4 \cdot \pi}$$

Nachklausur zur 2. Übungsklausur am 2.7.2009

Bsp 2

$$\boxed{x^2y'' - 3xy' + 7y = x^2\ln x} \qquad y(1) = 0 \quad y'(1) = 1$$

$x = e^t$ $\quad \boxed{t = \ln x}$

$y'_x = y'_t e^{-t}$

$$y''_x = (y''_t - y'_t)\, e^{-2t}$$

$$y''_t - y'_t - 3y'_t + 7y = t e^{2t}$$

$$k^2 - 4k + 7 = 0$$

$$D = 16 - 28 = -12$$

$$k = \frac{4 \pm \sqrt{-12}}{2} = 2 \pm \sqrt{3}\, i$$

Fund.Sys. $\quad e^{2t}\sin\sqrt{3}t \,,\ e^{2t}\cos\sqrt{3}t$

$$y_h = c_1 e^{2t}\sin\sqrt{3}t + c_2 e^{2t}\sin\sqrt{3}t$$

$$y_p = (At + B)\, e^{2t}$$

$$y_p' = Ae^{2t} + 2(At+B)e^{2t} = \underline{e^{2t}(2At + (A+B))}$$

$$y_p'' = 2(2At + (A+B))\, e^{2t} + e^{2t}\cdot 2A$$

$$= e^{2t}(4At + 2A + 2B + 2A) =$$

$$= e^{2t}\,\underline{(4At + 4A + 2B)}$$

$$y_p'' - 4y_p' + 7y_p =$$

$$= (4At + 4A + 2B)\, e^{2t} - 4e^{2t}(2At + (A+B)) + 7(At+B)e^{2t}$$

$$= t e^{2t}$$

$$e^{2t}(-4At + 7At - 2B + 7B) = t e^{2t}$$

$$3A = 1 \qquad 5B = 0 \quad \Rightarrow \boxed{B = 0} \qquad \boxed{A = \frac{1}{3}}$$

$y_p = \frac{t}{3} e^{2t}$

Allgemeine Lösung

$$\boxed{y = c_1 e^{2t} \sin(\sqrt{3}t) + c_2 e^{2t} \cos(\sqrt{3}t) + \frac{t}{3} e^{2t}}$$

$$\boxed{y = c_1 x^2 \sin(\sqrt{3}\ln x) + c_2 x^2 \cos(\sqrt{3}\ln x) + \frac{\ln x}{3} \cdot x^2}$$

Spezielle Lösung

$y(1) = 0 + c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

$$y'(x) = 2c_1 x \sin(\sqrt{3}\ln x) + c_1 x^2 \sqrt{3}\cos(\sqrt{3}\ln x) \cdot \frac{1}{\cancel{x}} +$$

$$+ 2c_2 x \cos(\sqrt{3}\ln x) - c_2 x^2 \sqrt{3}\sin(\sqrt{3}\ln x) \cdot \frac{1}{\cancel{x}} + \frac{x^2}{3x} + \frac{2x}{3}\ln x$$

$$y'(1) = \sqrt{3}c_1 + 2c_2 + \frac{1}{3} = 1$$

$$\sqrt{3}c_1 = \frac{2}{3} \Rightarrow \boxed{c_1 = \frac{2}{3\sqrt{3}}}$$

Lösung des Anfangswertproblems:

$$y_1(x) = \frac{2}{3\sqrt{3}} x^2 \sin(\sqrt{3}\ln x) + \frac{\ln x}{3} x^2$$

(3) Nachklausur zur 2. Übungsklausur am 2.7.09

Bsp. 3

$$\vec{x}(t) = \begin{pmatrix} t^2 \\ \frac{4}{3} t^{3/2} \\ t \end{pmatrix}$$

Begleitendes Dreibein in $P_1 = \begin{pmatrix} 1 \\ \frac{4}{3} \\ 1 \end{pmatrix}$

Länge des Kurvenstücks von $P_1 = \begin{pmatrix} 1 \\ 4/3 \\ 1 \end{pmatrix}$ bis $P_2(0,0,0)$

$$\dot{x} = \begin{pmatrix} 2t \\ 2t^{1/2} \\ 1 \end{pmatrix} \qquad \ddot{x} = \begin{pmatrix} 2 \\ t^{-1/2} \\ 0 \end{pmatrix} \qquad \dddot{x} = \begin{pmatrix} 0 \\ -\frac{1}{2} t^{-3/2} \\ 0 \end{pmatrix}$$

$$\|\dot{x}\| = \sqrt{4t^2 + 4t + 1} = \sqrt{(1+2t)^2} = 1 + 2t$$

(a) $$L = \int_0^1 (1+2t)\, dt = t + \frac{2t^2}{2} \Big|_0^1 = t + t^2 = 2$$

(b) begleitendes Dreibein

<u>normierter Tangentenvektor $\vec{v_1}$</u>

$$\vec{v_1} = \frac{\dot{x}(t)}{\|\dot{x}(t)\|} \qquad P_1 = \left(1, \frac{4}{3}, 1\right)^T$$

$$t = 1$$

$$\dot{\vec{x}}(1) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad \vec{v_1}(1) = \frac{\dot{x}(1)}{\|\dot{x}(1)\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \boxed{\vec{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

$$\dot{\vec{x}}(1) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \|\dot{x}(1)\| = \sqrt{5}$$

<u>Hauptnormalenvektor $\vec{v_2}$</u>

$$\vec{v_2} = \left(\ddot{\vec{x}} - \frac{\langle \dot{\vec{x}}, \ddot{\vec{x}} \rangle}{\|\dot{\vec{x}}\|^2} \dot{x} \right) \cdot \frac{1}{\| \ddot{\vec{x}} - \frac{\langle \dot{\vec{x}}, \ddot{\vec{x}} \rangle}{\|\dot{\vec{x}}\|^2} \dot{\vec{x}} \|}$$

$$\ddot{\vec{x}}(1) = \begin{pmatrix}2\\1\\0\end{pmatrix} \qquad \left\langle \dot{\vec{x}}(1), \ddot{\vec{x}}(1) \right\rangle = \left\langle \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix} \right\rangle = 6$$

$$\ddot{\vec{x}}(1) - \frac{\langle \dot{\vec{x}}(1), \ddot{\vec{x}}(1) \rangle}{\|\dot{x}(1)\|^2} \dot{\vec{x}}(1) = \begin{pmatrix}2\\1\\0\end{pmatrix} - \frac{6}{5}\begin{pmatrix}2\\2\\1\end{pmatrix} =$$

$$= \begin{pmatrix}-2/5\\-7/5\\-6/5\end{pmatrix} = -\frac{1}{5}\begin{pmatrix}2\\7\\6\end{pmatrix}$$

$$\vec{v}_2 = \frac{-1/5 \begin{pmatrix}2\\7\\6\end{pmatrix}}{\frac{1}{5}\sqrt{4+36+49}} = \boxed{\frac{-1}{\sqrt{89}}\begin{pmatrix}2\\7\\6\end{pmatrix} = \vec{v}_2}$$

Binormalenvektor $\vec{V}_3$

$$\vec{V}_3(1) = \vec{v}_1(1) \times \vec{v}(2)$$

$$\vec{V}_3(1) \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{1}{\sqrt{5}}2 & \frac{1}{\sqrt{5}}2 & \frac{1}{\sqrt{5}}1 \\ \frac{-1}{\sqrt{89}}2 & \frac{-1}{\sqrt{89}}7 & \frac{-1}{\sqrt{89}}6 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{89}}(-12+1) \\ \frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{89}}(-2+12) \\ \frac{2}{\sqrt{5}\sqrt{89}}(-7+2) \end{pmatrix} = \boxed{\frac{1}{\sqrt{445}}\begin{pmatrix}-11\\10\\-10\end{pmatrix} = \vec{V}_3}$$

(c) Schmiegebene

$$|x - x_0, \dot{x}, \ddot{x}| = 0$$

für $t = 1$

$$\begin{vmatrix} x-1 & 2 & 2 \\ y-\frac{4}{3} & 2 & 1 \\ z-1 & 1 & 0 \end{vmatrix} = 0 \Rightarrow -(x-1) + 2(z-1) + 2\left(y - \frac{4}{3} - 2z + 2\right) = 0$$

$$-x + 1 + 2z - 2 + 2y - \frac{8}{3} - 4z + 4 = 0$$

$$\boxed{-x + 2y - 2z = -\frac{1}{3}}$$

Normalebene

$$\langle X - x_0 , \dot{x} \rangle = 0$$

für $t = 1$

$$\left\langle \begin{pmatrix} x-1 \\ y-\frac{4}{3} \\ z-1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle = 0$$

$$2(x-1) + 2\left(y-\frac{4}{3}\right) + (z-1) = 0$$

$$\boxed{2x + 2y + z - 5\frac{1}{3} = 0}$$

Streckebene

$$\left\langle X - x_0 , \underbrace{\ddot{x} - \frac{\dot{x}\,\ddot{x}}{\|\dot{x}\|^2}\,\dot{x}}\right\rangle = 0$$

für $t = 1$ aus (b) $= -\frac{1}{5}\begin{pmatrix} 2 \\ 7 \\ 6 \end{pmatrix}$

$$\left\langle \begin{pmatrix} x-1 \\ y-\frac{4}{3} \\ z-1 \end{pmatrix}, \begin{pmatrix} -\frac{2}{5} \\ -\frac{7}{5} \\ -\frac{6}{5} \end{pmatrix} \right\rangle = 0$$

$$-\frac{2}{5}(x-1) - \frac{7}{5}\left(y-\frac{4}{3}\right) - \frac{6}{5}(z-1) = 0$$

$$-2x + 2 - 7y + \frac{28}{3} - 6z + 6 = 0$$

$$\boxed{-2x - 7y - 6z + 17\frac{1}{3} = 0}$$