Gruppe B

(1) $A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 2 & 1 \\ -2 & -1 & -4 \end{pmatrix}$

Eigenwerte

$$\begin{vmatrix} 1-\lambda & 0 & 4 \\ 2 & 2-\lambda & 1 \\ -2 & -1 & -4-\lambda \end{vmatrix} = (1-\lambda)\big((2-\lambda)(-4-\lambda)+1\big) + 4\big(-2+2(2-\lambda)\big) =$$

$$= (1-\lambda)\left(\lambda^2+2\lambda-8+1\right) + 4(4-2-2\lambda)$$

$$= (1-\lambda)\left(\lambda^2+2\lambda-7\right) + 8(1-\lambda) =$$

$$= (1-\lambda)\left(\lambda^2+2\lambda+1\right) = (1-\lambda)(\lambda+1)^2 = 0$$

$\lambda_1 = 1$, $\lambda_2 = -1$ zweifach

Algebraische Vielfachheit von $\lambda_1 = 1$ ist 1
— " — $\lambda_2 = -1$ ist 2

Eigenvektoren zu $\lambda_1 = 1$

$$\begin{pmatrix} 1-1 & 0 & 4 \\ 2 & 2-1 & 1 \\ -2 & -1 & -4-1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Matrix Transformation: Vertausch 1. Z mit 2. Z.
$3.Z \leftarrow 3.Z + 1.Z$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3.Z \leftarrow 3.Z + 2.Z$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$y = t \in \mathbb{R}$

$4z = 0 \Rightarrow z = 0$

$2x + y + z = 0 \Rightarrow x = -\frac{y}{2} = -\frac{t}{2}$

Eigenvektoren $\left\{ \begin{pmatrix} -t/2 \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$

Geometrische Vielfachheit $= 1$

Eigenvektoren zu $\lambda_2 = -1$

$$\begin{pmatrix} 1+1 & 0 & 4 \\ 2 & 2+1 & 1 \\ -2 & -1 & -4+1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2.Z \leftarrow 2.Z - 1.Z \,;\quad 3.Z \leftarrow 3.Z + 1.Z$

$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & -3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$3.Z \leftarrow 3.Z + \frac{1}{3}\, 2.Z$

$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$z = t \in \mathbb{R}$

$3y - 3z = 0 \Rightarrow y = z = t$

$2x + 4z = 0 \Rightarrow x = -2z = -2t$

Eigenvektoren: $\left\{ \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$

Geometrische Vielfachheit $= \underline{1}$

2a) $f(x y) = \cos(xy) + x \ln(y^2)$ $\qquad f(\pi, 1) = -1$

$(x_0 y_0) = (\pi, 1)$

$f_x = -y \sin(xy) + \ln(y^2)$ $\qquad f_x(\pi, 1) = 0$

$f_y = -x \sin(xy) + 2xy \frac{1}{y^2} = x \sin(xy) + 2\frac{x}{y}$ $\qquad f_y(\pi, 1) = 2\pi$

$f_{xx} = -y^2 \cos(xy)$ $\qquad f_{xx}(\pi, 1) = 1$

$f_{xy} = -\sin(xy) - y\cancel{x}\cos(xy) + \frac{1}{y^2} \cdot 2y$ $\qquad f_{xy}(\pi, 1) = \pi + 2$

$= -\sin(xy) - xy\cos(xy) + \frac{2}{y}$

$f_{yy} = -x^2 \cos(xy) - \frac{2x}{y^2}$ $\qquad f_{yy}(\pi, 1) = \pi^2 - 2\pi$

$$T(xy) = f(x_0 y_0) + f_x(x_0 y_0)(x - x_0) + f_y(x_0 y_0)(y - y_0)$$
$$+ \frac{1}{2}\left[f_{xx}(x_0 y_0)(x - x_0)^2 + 2 f_{xy}(x_0 y_0)(x - x_0)(y - y_0) + f_{yy}(y - y_0)^2 \right]$$

$$T(xy) = -1 + 2\pi(y - 1) + \frac{1}{2}\left[(x - \pi)^2 + 2(\pi + 2)(x - \pi)(y - 1) + (\pi^2 - 2\pi)(y - 1)^2 \right]$$

2b $\qquad f(3.15, 1.1) = -0.3477$

$$T(3.15, 1.1) = -1 + 2\pi \cdot 0.1 + \frac{(0.01)^2}{2} + (\pi + 2)\, 0.1 \cdot 0.01 + \frac{(\pi^2 - 2\pi)(0.1)^2}{2}$$

$T(3.15, 1.1) = -0.3486$

Abs. Fehler $= T(3.15, 1.1) - f(3.15, 1.1) = 0.0007$

Rel Fehler $= \left| \frac{T(3.15, 1.1) - f(3.15, 1.1)}{f(3.15, 1.1)} \right| = 0.2\,\%$

(3) $f(x,y) = \frac{xy}{64} - \frac{1}{x} + \frac{1}{y}$ $\quad x \neq 0 \quad y \neq 0$

Lokale Extreme

$f_x = \frac{y}{64} + \frac{1}{x^2} = 0$ $\quad x^2 y = -64$

$f_y = \frac{x}{64} - \frac{1}{y^2} = 0$ $\quad xy^2 = 64 \Rightarrow x = \frac{64}{y^2}$

$\left(\frac{64}{y^2}\right)^2 y = -64$

$y^3 = -64 \Rightarrow \boxed{y = -4}$

$x = \frac{64}{16} = 4$

kritische Punkte

$P(4,-4)$

$f_{xx} = -\frac{2}{x^3}$

$f_{xy} = \frac{1}{64} = f_{yx}$ $\quad H_f(x,y) = \begin{pmatrix} -2/x^3 & 1/64 \\ \frac{1}{64} & \frac{2}{y^3} \end{pmatrix}$

$f_{yy} = \frac{2}{y^3}$

$H_f(4,-4) = \begin{pmatrix} -2/64 & 1/64 \\ 1/64 & \frac{2}{-64} \end{pmatrix}$

$\det H_f(4,-4) = \frac{4}{64^2} - \frac{1}{64^2} > 0$ $\Rightarrow$ $P(4,-4)$ ist ein lok. Maximum

$-\frac{2}{64} < 0$

$f(4,-4) = -\frac{3}{4}$

- Globale Extreme $\quad 1 \leq x \leq 9 \quad -9 \leq y \leq -1$

$\boxed{x = 1 \quad -9 \leq y \leq -1}$

$f(1,y) = \frac{y}{64} - 1 + \frac{1}{y}$

$f'(1,y) = \frac{1}{64} - \frac{1}{y^2} = 0 \Rightarrow y = \pm 8$ $\Rightarrow$ ausgeschl.

$y=-8$ kritischer Punkt
$f''(1,y) = \frac{2}{y^3} < 0$ $\Rightarrow$ $y=-8$ lok. Max. am Rand.

$x = 9 \quad -1 \le y \le 9$

$f(9,y) = \frac{9y}{64} - \frac{1}{y} + \frac{1}{y}$

$f'(9,y) = \frac{9}{64} - \frac{1}{y^2} = 0 \Rightarrow y = \pm\frac{8}{3}$ ausgeschl.

$f''(9,y) = \frac{2}{y^3} < 0$ $\Rightarrow$ $y = \frac{8}{3}$ kein lok. Max am Rande

$y = -1 \quad 1 \le x \le 9$

$f(x,-1) = -\frac{x}{64} - \frac{1}{x} - 1$

$f'(x,-1) = -\frac{1}{64} + \frac{1}{x^2} = 0 \Rightarrow x = \pm 8$, ausgeschl.

$f''(x,-1) = -\frac{2}{x^3} < 0$

$\Rightarrow$ $x = 8$ lok. Max am Rande

$y = -9 \quad 1 \le x \le 9$

$f(x,-9) = \frac{-9x}{64} - \frac{1}{x} - \frac{1}{9}$

$f'(x,-9) = -\frac{9}{64} + \frac{1}{x^2} = 0 \Rightarrow x = \pm\frac{8}{3}$, ausgeschl.

$f''(x,-9) = -\frac{2}{x^3} < 0$

$\Rightarrow$ $x = \frac{8}{3}$ lok. Max am Rande

Kandidaten Extrema (global)

$P_1(4,-4)$	$f(4,-4) = -3/4$	$P_6(1,-1)$	$f(1,-1) = -2.0156$
$P_2(1,-8)$	$f(1,-8) = -5/4$	$P_7(1,-9)$	$f(1,-9) = -1.2517$
$P_3(9,-\frac{8}{3})$	$f(9,-\frac{8}{3}) = -0.8611$	$P_8(9,-1)$	$f(9,-1) = -1.4878$
$P_4(8,-1)$	$f(8,-1) = -1.25$	$P_9(9,-9)$	$f(9,-9) = -1.2517$
$P_5(\frac{8}{3},-9)$	$f(8/3,-9) = -0.8611$		

glob. Max $P_1(4,-4)$ mit Wert -0.75; glob Min $P_6(1,-1)$ Wert -2.0156