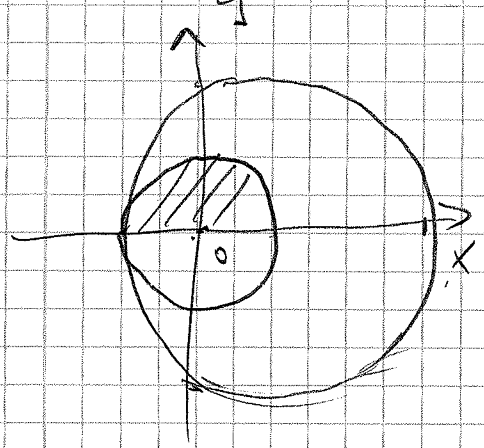
Gruppe A Lösung 1. Beispiel

2. Klausur 18.6.08

$$G = \{ (x y z) \quad (x-1)^2 + y^2 + z^2 \leq 4 \quad x^2 + y^2 \leq 1$$
$$z \geq 0 \quad y \geq 0$$



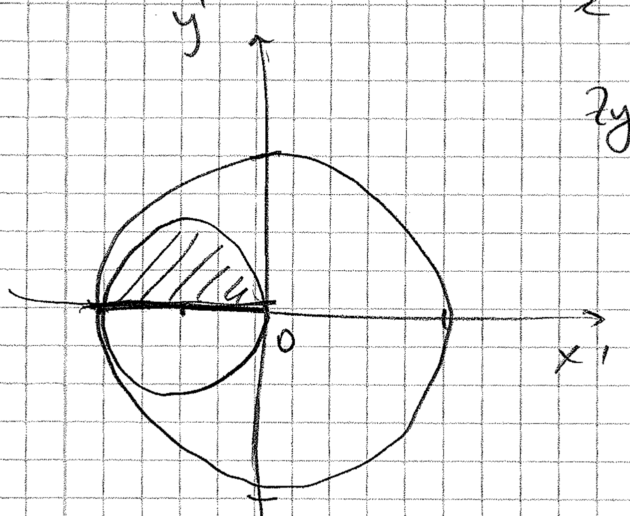
Koord. transformation

$x' = x - 1$

$y' = y \qquad |J| = 1$

$z' = z$

$$G' = \{ (xyz) : \quad x'^2 + y'^2 + z'^2 \leq 4 \quad (x'+1)^2 + y^2 \leq 1 \}$$
$$z' \geq 0 \quad y' \geq 0$$



Zylinderkoordinaten

$x' = r\cos\varphi$

$y' = r\sin\varphi$

$z' = z$

$dV = r\,dr\,d\varphi\,dz$

Gleichungen des gegebenen Bereichs in den neuen Koordinaten:

$\boxed{r^2 + z^2 \leq 4}$ $\qquad (r\cos\varphi + 1)^2 + r^2\sin^2\varphi \leq 1$

$r^2\cos^2\varphi + r^2\sin^2\varphi + 2r\cos\varphi \leq 0$

$r^2 + 2r\cos\varphi \leq 0$

$\boxed{r \leq -2\cos\varphi} \Rightarrow \cos\varphi < 0 \Rightarrow \varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$\boxed{z \geq 0}$ $\quad y \geq 0$

$\sin\varphi \geq 0 \Rightarrow \boxed{\varphi \in [0, \pi]}$

Integrationsgrenzen

$$\varphi \in \left[\frac{\pi}{2}, \pi\right]$$

$$r \in [0, -2\cos\varphi)$$

$$z \in [0, \sqrt{4-r^2}$$

$$V = \int_{\frac{\pi}{2}}^{\pi} \int_0^{-2\cos\varphi} \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^{-2\cos\varphi} \sqrt{4-r^2} \, r \, dr \, d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^{-2\cos\varphi} -\frac{1}{2} \sqrt{4-r^2} \; d(-r^2) \; d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\pi} -\frac{1}{2} \left. \frac{(4-r^2)^{3/2}}{\frac{3}{2}} \right|_0^{-2\cos\varphi} d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\pi} -\frac{1}{2} \left[\frac{(4-4\cos^2\varphi)^{3/2}}{\frac{3}{2}} - \frac{4^{3/2}}{\frac{3}{2}} \right] d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\pi} -\frac{1}{\cancel{2}} \frac{\cancel{2}}{3} \left[(2\sin\varphi)^3 - 8 \right] d\varphi = -\frac{8}{3} \int_{\frac{\pi}{2}}^{\pi} (\sin^3\varphi - 1) \, d\varphi$$

$$= -\frac{8}{3} \left[\int_{\frac{\pi}{2}}^{\pi} \sin^3\varphi \, d\varphi - \int_{\frac{\pi}{2}}^{\pi} d\varphi \right] = \frac{8}{3} \left(\frac{\pi}{2} - \underbrace{\int_{\frac{\pi}{2}}^{\pi} \sin^3\varphi \, d\varphi}_{=: I} \right)$$

$$\int \sin^3\varphi \, d\varphi = \int (1-\cos^2\varphi)\, d(-\cos\varphi) =$$

$$= \int (\cos^2\varphi - 1)\, d\cos\varphi = \frac{\cos^3\varphi}{3} - \cos\varphi$$

$$I = \int_{\frac{\pi}{2}}^{\pi} \sin^3\varphi \, d\varphi = \frac{\cos^3\varphi}{3} - \cos\varphi \Big|_{\frac{\pi}{2}}^{\pi} = \cos\frac{\pi}{2} - \frac{\cos^3\frac{\pi}{2}}{3}$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

Somit (in V eingesetzt):

$$V = \frac{r}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) \quad //$$

Gruppe A BSP 2.

2. Klausur 19.6.2009

a) $\underbrace{(1-y^2x)}_{A}\,dx \underbrace{-(2yx^2+xe^y)}_{B}\,dy = 0$

$$A_y = -2yx \qquad B_x = -4yx + e^y$$

$A_y \neq B_x \Rightarrow$ nicht exakt.

$\exists$ Integr. Faktor?

$$\frac{A_y - B_x}{B} = \frac{-2yx + 4yx - e^y}{-2yx^2 + xe^y} = \frac{\cancel{2yx - e^y}}{-x(\cancel{2yx - e^y})}$$

$= f(x)$ Fukt. von nur einer Variablen

Integr. Faktor: $\mu(x) = \exp\left(\int \frac{A_y - B_x}{B}\,dx\right)$

$$\mu(x) = \exp(-\ln|x| + c) = \tilde{c}/|x|$$

für $x \neq 0$

Wähle einen Integr. Faktor aus dieser Klasse $\mu(x) = \frac{1}{x}$ und multipliziere beide Seiten der Gleichung damit:

$$\underbrace{\frac{1-y^2x}{x}}_{A'}\,dx - \underbrace{\frac{2yx^2 + xe^y}{x}}_{B'}\,dy = 0$$

$A'_y = -2y \quad B'_x \; -2y$ ✓

Lösungsansatz $F(xy) = K$ Konstante

$$F_x = A' \qquad F_y = B'$$

$$\Downarrow$$

$$F = \int A'(x)\,dx + \varphi(y)$$

$$F = \int \frac{1 - y^2 x}{x}\,dx + \varphi(y) = \ln|x| - y^2 x + \varphi(y)$$

$$F_y' = \cancel{-2yx} + \varphi'(y) = B' = \cancel{-2yx} - e^y$$

$$\varphi'(y) = -e^{+y}$$

$$\Downarrow$$

$$\varphi(y) = -\int e^y\,dy = -e^y$$

Lösung: $\ln|x| - y^2 x - e^y = K$ Konstante

1(b) $y(1) = 1$

Setze $(x,y) = (1,1)$ in die allg. Lösung ein und bestimme Konstante:

$$K = -1 - e = -(e+1)$$

Spezielle Lösung $\ln|x| - y^2 x - e^y = -(e+1)$

Gruppe A 3. Beispiel

2. Klausur 19.6.09

$$\vec{X}(t) = \begin{pmatrix} \sin(2t) \\ \cos(2t) \\ t^2 + 28 \end{pmatrix}$$

$$\dot{\vec{X}}(t) = \begin{pmatrix} 2\cos(2t) \\ -2\sin(2t) \\ 2t \end{pmatrix}$$

$$\|\dot{X}(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t + 4t^2} = 2\sqrt{1+t^2}$$

$P_1 = (0, 1, 28)$ entspricht $t_1 = 0$

$P_2 = (1, 0, \frac{\pi^2}{16} + 28)$ entspricht $t_2 = \frac{\pi}{4}$

$$L = \int_0^{\frac{\pi}{4}} \|\dot{\vec{X}}(t)\|\, dt = \int_0^{\frac{\pi}{4}} 2\sqrt{1+t^2}\, dt = 2\int_1^{a} \cosh^2 u\, du$$

$\underline{a = \cosh\left(\frac{\pi}{4}\right)}$ $\underline{1 = \cosh(0)}$

Sei $I = \int \sqrt{1+t^2}\, dt = \underline{\int \cosh^2 u\, du}$ → partiell mit $x' = \cosh u$, $y = \cosh u$

$t = \sinh u$ $dt = \cosh u\, du$

$\sqrt{1+t^2} = \cosh u$

$$= \cosh u \sinh u - \int \sinh^2 u\, du =$$

$$= \cosh u \sinh u - \int (\cosh^2 u - 1)\, du = \cosh u \sinh u + u - \int \cosh^2 u\, du$$

Aus $\int \cosh^2 u\, du = \cosh u \sinh u + u - \int \cosh^2 u\, du$

folgt $\int \cosh^2 u\, du = \dfrac{\cosh u \sinh u + u}{2}$

$$L = 2\, \frac{\cosh u \sinh u + u}{2}\Bigg|_1^a = \boxed{\cosh a \sinh a + a - \cosh 1 \sinh 1 - 1}$$

b) Krümmung $k(t) = ?$

$$\dddot{\vec{X}} = \begin{pmatrix} -4 \sin(2t) \\ -4 \cos(2t) \\ 2 \end{pmatrix}$$

$$k = \frac{1}{\|\dot{X}\|^3} \sqrt{\|\dot{x}\|^2 \|\ddot{x}\|^2 - \langle \dot{x}, \ddot{x} \rangle}$$

$$\|\ddot{X}\|^2 = 16 \sin^2(2t) + 16 \cos^2(2t) + 4 = 16+4 = 20$$

$$\langle \dot{X}, \ddot{X} \rangle = \left\langle \begin{pmatrix} 2 \cos(2t) \\ -2 \sin(2t) \\ 2t \end{pmatrix} \begin{pmatrix} -4 \sin(2t) \\ -4 \cos(2t) \\ 2 \end{pmatrix} \right\rangle =$$

$$= -8 \cos(2t) \sin(2t) + 8 \sin(2t) \cos(2t) + 4t = 4t.$$

$$K(t) = \frac{1}{8(1+t^2)\sqrt{1+t^2}} \sqrt{4(1+t^2) \cdot 20 - 16t^2}$$

$$k(t) = \sqrt{5+4t^2}$$

e) Torsion für $t = 2$

$$\tau = \frac{(\dot{x} \ \ddot{x} \ \dddot{x})}{\|\dot{x}\|^2 \|\ddot{x}\|^2 - \langle \dot{x}, \ddot{x} \rangle^2}$$

$$\dddot{\vec{X}} = \begin{pmatrix} -8 \cos(2t) \\ 8 \sin(2t) \\ 0 \end{pmatrix}$$

$$(\dot{x}, \ddot{x}, \dddot{x}) = \begin{vmatrix} 2 \cos(2t) & -4 \sin(2t) & -8 \cos(2t) \\ -2 \sin(2t) & -4 \cos(2t) & 8 \sin(2t) \\ 2t & 2 & 0 \end{vmatrix} =$$

$$(\dot{X}, \ddot{X}, \dddot{X}) = 2t \begin{vmatrix} -4\sin(2t) & -8\cos(2t) \\ -4\cos(2t) & 8\sin(2t) \end{vmatrix} - 2 \begin{vmatrix} 2\cos(2t) & -8\cos(2t) \\ -2\sin(2t) & 8\sin(2t) \end{vmatrix}$$

$$= 2t(-32) = -64t$$

$$\tau(2) = \frac{-64 \cdot 2}{2^2(1+2^2) \cdot 20 - (4 \cdot 2)^2} = \frac{-128}{400-64}$$

$$\tau(2) = -\frac{128}{336} \neq -0.3810$$