Combinatorial optimization 1
Winter term 2016/2017

7th working sheet

43. Let $A \in \mathbb{R}^{m \times n}$, $m, n \in \mathbb{N}$, be a unimodular matrix. Are the following matrices also unimodular: $-A$, A^T, $(A|A)$, A^{-1}, $A|I_m$? I_m denotes the $m \times m$ identity matrix, A^T is the transposed of A, and the operator $|$ denotes the column-wise concatenation of matrices.

44. Consider the following statement about total unimodular matrices (the theorem of Ghouila-Houri, 1962): A matrix $A = (a_{ij}) \in \{0, 1, -1\}^{m \times n}$ is total unimodular, if and only if for any subset $R \subseteq \{1, 2, \dots, m\}$ of the row indices there is a partition $R = R_1 \uplus R_2$ such that

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\},$$

for all $j = 1, 2, \dots, n$ [1].

(a) Show that the above theorem of Ghouila-Houri implies the following statement, known as the theorem of Heller and Tompkins, 1956.

Let A be an m by n matrix with a partition of its row indices into two disjoint sets B and C. Then the following four conditions together are sufficient for A to be totally unimodular:

(i) Every column of A contains at most two non-zero entries;

(ii) Every entry in A equals 0, +1, or 1;

(iii) If two non-zero entries in a column of A have the same sign, then the row of one of these entries is in B and the row of the other entry is in C;

(iv) If two non-zero entries in a column of A have opposite signs, then the rows of the two entries are either both in B or both in C.

Moreoever show that a total unimodular matrix $A = (a_{ij}) \in \{0, -1, +1\}^{m \times n}$ with at most two non-zero entries per column fulfills conditions *(iii)* and *(iv)* above, thus these conditions are necessary for the total unimodularity of such a matrix A.

(b) Is the following matrix total unimodular?

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(c) Show by means of a counterexample that the total unimodularity of matrices A und B is not sufficient for the total unimodularity of the matrix $(A|B)$.

45. An $m \times n$ $0-1$ matrix is called *interval matrix* if the 1-entries of any column j, $j \in \{1, 2, \dots, n\}$, build an interval, i.e. $a_{ij} = a_{kj} = 1$ with $k > i + 1$ implies $a_{\ell j} = 1$ for all $\ell = i + 1, \dots, k - 1$. Show that the interval matrices are total unimodular.

[1] You don't need to prove this theorem, but you can of course try to do so. The proof can be found e.g. in *B. Korte und J. Vygen, Combinatorial Optimization: Theory and Algorithms, Springer, Fifth Edition, 2012, pp. 114–115.*